

Lecture 1 :Determinants and solution of linear equation by Gramer's rule

Introduction

A linear equation is an equation that represents a mathematical relationship between one or more variables, where the highest power of the variable is one. It is called "linear" because its graph is a straight line.

General Form of a Linear Equation

For one variable: $ax + b = 0$

Example: $2x + 6 = 0$

For two variables: $ax + by = c$

Example: $3x + 2y = 6$

Methods of Solving

1. For one variable:

Example: $2x + 6 = 0 \rightarrow x = -3$

2. For two variables:

Example:

$$2x + y = 5$$

$$x - y = 1$$

Solution: $(x, y) = (2, 1)$

2- Determinant is a numerical value that can be calculated from a square matrix. It plays a key role in solving systems of linear equations, finding matrix inverses, computing areas and volumes, and analyzing linear transformations.

***Definition**

For a square matrix A of order $n \times n$, the determinant is denoted by $|A|$ or $\det(A)$. The determinant provides information about whether the matrix is invertible and its scaling factor.

*** Determinant of a 2×2 Matrix**

If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ then $|A| = ad - bc$

Example 1:

$$A = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \quad \rightarrow \quad |A| = (3 \times 4) - (2 \times 1) = 10.$$

Example 2:

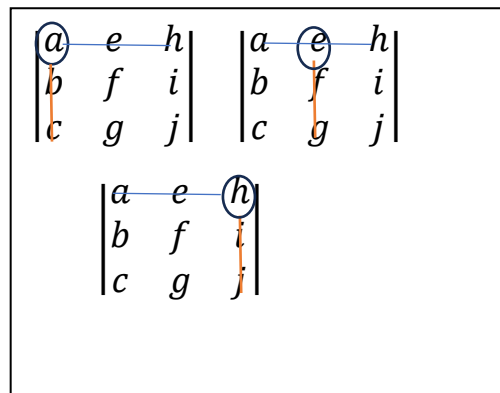
$$A = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \quad \rightarrow \quad |A| = (2 \times 2) - (-1 \times 1) = 5.$$

Example 3:

$$A = \begin{vmatrix} 4 & 2 \\ 1 & -1 \end{vmatrix} \quad \rightarrow \quad |A| = (4 \times -1) - (2 \times 1) = -6.$$

*** Determinant of a 3×3 Matrix**

If
$$A = \begin{vmatrix} a & e & h \\ b & f & i \\ c & g & j \end{vmatrix}$$



$$\Delta = a \cdot \begin{vmatrix} f & i \\ g & j \end{vmatrix} - e \cdot \begin{vmatrix} b & i \\ c & j \end{vmatrix} + h \cdot \begin{vmatrix} b & f \\ c & g \end{vmatrix}$$

$$\Delta = -b \cdot \begin{vmatrix} e & h \\ g & j \end{vmatrix} + f \cdot \begin{vmatrix} a & h \\ c & j \end{vmatrix} - i \cdot \begin{vmatrix} a & e \\ c & g \end{vmatrix}$$

$$\Delta = h \cdot \begin{vmatrix} b & f \\ c & g \end{vmatrix} - i \cdot \begin{vmatrix} a & e \\ c & g \end{vmatrix} + j \cdot \begin{vmatrix} a & e \\ b & f \end{vmatrix}$$

In this method, the following must be taken into consideration :

When selecting the element for which the auxiliary determinant is to be found, the necessary symbol must be placed in the partition, as follows:

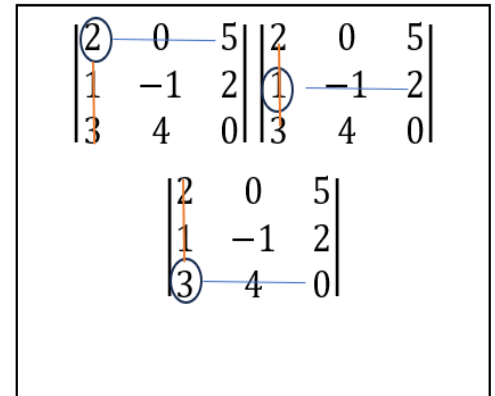
$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Example 1: find the determinant for the following matrix :

$$A = \begin{vmatrix} 2 & 0 & 5 \\ 1 & -1 & 2 \\ 3 & 4 & 0 \end{vmatrix}$$

$$f|A| = 2((-1 \times 0) - (4 \times 2)) - 1((0 \times 0) - (4 \times 5)) + 3((0 \times 2) - (-1 \times 5))$$

$$|A| = -16 + 20 + 15 = 19$$



Example 2: find the determinant for the following matrix :

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$\begin{aligned} |A| &= 1((5 \times 9) - (6 \times 8)) - 2((4 \times 9) - (6 \times 7)) + 3((4 \times 8) - (5 \times 7)) \\ &= -3 + 12 - 9 \\ &= 0 \end{aligned}$$

Example 3: find the determinant for the following matrix :

$$A = \begin{vmatrix} 3 & -2 & 1 \\ 0 & 4 & -3 \\ 2 & 5 & 6 \end{vmatrix}$$

$$\begin{aligned} |A| &= 3(4 \times 6 - (-3) \times 5) - (-2)(0 \times 6 - (-3) \times 2) + 1(0 \times 5 - 4 \times 2) \\ &= 3(24 + 15) + 2(0 + 6) + 1(0 - 8) \\ &= 3(39) + 12 - 8 = 117 + 12 - 8 = 121 \end{aligned}$$

Example 3: find the determinant for the following matrix :

$$A = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 5 & 2 \end{vmatrix}$$

$$\begin{aligned} |A| &= 2(3 \times 2 - 4 \times 5) - 0(-1 \times 2 - 4 \times 0) + 1(-1 \times 5 - 3 \times 0) \\ &= 2(6 - 20) - 0(-2) + 1(-5) = 2(-14) + 0 - 5 \\ &= -28 - 5 = -33 \end{aligned}$$

$$|A| = -33$$

* Properties of Determinants

1. If two rows (or columns) are equal \rightarrow determinant = 0.
2. Swapping two rows (or columns) changes the sign of the determinant.
3. Multiplying a row by k multiplies the determinant by k.
4. The determinant of a triangular matrix equals the product of its diagonal elements.
5. $\det(A^{-1}) = 1 / \det(A)$.
6. If $\det(A) = 0$, the matrix is singular (not invertible).

2- Solution of linear equation by (Cramer's rule)

The value of each unknown is the result of dividing two determinants, one in the numerator and the other in the denominator. The determinant written in the denominator is the same as the determinant of the denominator, after changing the factors of the unknown whose value is to be found in the same order.

Example 1 : solve the linear equation by using Cramer rule

$$\begin{aligned} 3x - 2y &= 8 \\ -5x + 4y &= -3 \end{aligned}$$

Sol:

$$1- \quad \begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}$$

$$D = 3 \times 4 - ((-5) \times (-2)) = 2$$

$$2- D_x = \begin{vmatrix} 8 & -2 \\ -3 & 4 \end{vmatrix}$$

$$D_x = 8*4 - (-2*-3) = 26$$

$$x = \frac{D_x}{D} = \frac{26}{2} = 13$$

$$3- D_y = \begin{vmatrix} 3 & 8 \\ -5 & -3 \end{vmatrix}$$

$$D_y = 3*(-3) - ((-5)* 8) = 31$$

$$y = \frac{D_y}{D} = \frac{31}{2} = 15.5$$

$$x = 13, \quad y = 15.5$$

Example2 : solve the linear equation by using Gramer rule

$$2x + 3y = 8$$

$$x - y = 1$$

$$D = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}$$

$$D = (2)(-1) - (3)(1) = -5$$

Dx: Replace the x-column with constants [8, 1]

$$D_x = \begin{vmatrix} 8 & 3 \\ 1 & -1 \end{vmatrix}$$

$$D_x = (8)(-1) - (3)(1) = -11$$

Dy: Replace the y-column with constants [8, 1]

$$D_y = \begin{vmatrix} 2 & 8 \\ 1 & 1 \end{vmatrix}$$

$$D_y = (2)(1) - (8)(1) = -6$$

$$x = \frac{D_x}{D} = \frac{-11}{-5} = 2.2$$

$$y = \frac{D_y}{D} = \frac{-6}{-5} = 1.2$$

$$x = 2.2, \quad y = 1.2$$

Example 3 : solve the linear equation by using Gramer rule

$$3x - 2y = 7$$

$$x + 4y = 10$$

$$D = \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix}$$

$$D = (3)(4) - (-2)(1) = 12 + 2 = 14$$

$$D_x = \begin{vmatrix} 7 & -2 \\ 10 & 4 \end{vmatrix}$$

$$D_x = (7)(4) - (-2)(10) = 28 + 20 = 48$$

$$D_y = \begin{vmatrix} 3 & 7 \\ 1 & 10 \end{vmatrix}$$

$$D_y = (3)(10) - (7)(1) = 30 - 7 = 23$$

$$x = \frac{D_x}{D} = \frac{48}{14} = 3.43$$

$$y = \frac{D_y}{D} = \frac{23}{14} = 1.64$$

$$x = 3.43, y = 1.64$$

Example 4 : solve the linear equation by using Gramer rule

$$5x + 2y = 12$$

$$3x - 4y = -6$$

$$D = \begin{vmatrix} 5 & 2 \\ 3 & -4 \end{vmatrix}$$

$$D = (5)(-4) - (2)(3) = -20 - 6 = -26$$

$$D_x = \begin{vmatrix} 12 & 2 \\ -6 & -4 \end{vmatrix} \rightarrow D_x = (12)(-4) - (2)(-6) = -48 + 12 = -36$$

$$D_y = \begin{vmatrix} 5 & 12 \\ 3 & -6 \end{vmatrix} \rightarrow D_y = (5)(-6) - (12)(3) = -30 - 36 = -66$$

$$x = \frac{D_x}{D} = \frac{-36}{-26} = 1.38$$

$$y = \frac{D_y}{D} = \frac{-66}{-26} = 2.54$$

Example 4: solve the linear equation by using Gramer rule

$$x + y + 2z = 6$$

$$2x + y + z = 4$$

$$x + 2y + 3z = 8$$

$$D = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\begin{aligned} D &= 1 * ((1*3) - (1*2)) - 1 * ((2*3) - (1*1)) + 2 * ((2*2) - (1*1)) \\ &= 1 - 5 + 6 = 2 \end{aligned}$$

$$D_x = \begin{vmatrix} 6 & 1 & 2 \\ 4 & 1 & 1 \\ 8 & 2 & 3 \end{vmatrix}$$

$$\begin{aligned} D_x &= 6 * ((1*3) - (1*2)) - 1 * ((4*3) - (1*8)) + 2 * ((4*2) - (1*8)) \\ &= 6 - 4 + 0 + 2 \end{aligned}$$

$$x = \frac{D_x}{D} = \frac{2}{2} = 1$$

$$D_y = \begin{vmatrix} 1 & 6 & 2 \\ 2 & 4 & 1 \\ 1 & 8 & 3 \end{vmatrix}$$

$$\begin{aligned} D_y &= 1 * ((4*3) - (1*8)) - 6 * ((2*3) - (1*1)) + 2 * ((2*8) - (4*1)) \\ &= 4 - 30 + 24 = -2 \end{aligned}$$

$$y = \frac{D_y}{D} = \frac{-2}{2} = -1$$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 4 \\ 1 & 2 & 8 \end{vmatrix}$$

$$\begin{aligned} D_z &= 1 * ((1*8) - (2*4)) - 1 * ((2*8) - (4*1)) + 6 * ((2*2) - (1*1)) \\ &= 0 - 12 + 18 \\ &= 6 \end{aligned}$$

$$z = \frac{D_z}{D} = \frac{6}{2} = 3$$

$$x = 1, \quad y = -1, \quad z = 3$$

Example 5: solve the linear equation by using Gramer rule

$$x + y + z = 6$$

$$2x - y + z = 3$$

$$x + 2y - z = 4$$

$$1- D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$\begin{aligned} D &= 1 * ((-1*-1) - (1*2)) - 1 * ((2*-1) - (1*1)) + 1 * ((2*2) - (-1*1)) \\ &= -1 + 3 + 5 = 7 \end{aligned}$$

$$2- D_x = \begin{vmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

$$\begin{aligned} D_x &= 6 * ((-1*-1) - (1*2)) - 1 * ((3*-1) - (1*4)) + 1 * ((3*2) - (-1*4)) \\ &= -6 + 7 + 10 = 11 \end{aligned}$$

$$x = \frac{D_x}{D} = \frac{11}{7} = 1.57$$

$$3- D_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & -1 \end{vmatrix}$$

$$D_y = 1 * ((3 * -1) - (1 * 4)) - 6 * ((2 * -1) - (1 * 1)) + 1 * ((2 * 4) - (1 * 3)) \\ = -7 + 18 + 5 = 16$$

$$y = \frac{D_y}{D} = \frac{16}{7} = 2.29$$

$$4- D_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$$

$$D_z = 1 * ((-1 * 4) - (3 * 2)) - 1 * ((2 * 4) - (3 * 1)) + 6 * ((2 * 2) - (-1 * 1)) \\ = -10 - 5 + 30 = 15$$

$$z = \frac{D_z}{D} = \frac{15}{7} = 2.14$$

$$x = 1.57, y = 2.29, z = 2.14$$

Example 6 : solve the linear equation by using Gramer rule

$$x + 2y + 3z = 14$$

$$2x - y + z = 3$$

$$3x + y + 2z = 10$$

$$1- D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$D = 1(-1 \times 2 - 1 \times 1) - 2(2 \times 2 - 1 \times 3) + 3(2 \times 1 - (-1) \times 3) \\ = 1(-3) - 2(1) + 3(5) = -3 - 2 + 15 = 10$$

$$2- D_x = \begin{vmatrix} 14 & 2 & 3 \\ 3 & -1 & 1 \\ 10 & 1 & 2 \end{vmatrix}$$

$$D_x = 14 * ((-1 * 2) - (1 * 1)) - 2 * ((3 * 2) - (1 * 10)) + 3 * ((3 * 1) - (-1 * 10)) \\ = (14 * -3) - (2 * -4) + (3 * 13) \\ = -42 + 8 + 39 = 5$$

$$x = \frac{D_x}{D} = \frac{5}{10} = 0.5$$

$$D_y = \begin{vmatrix} 1 & 14 & 3 \\ 2 & 3 & 1 \\ 3 & 10 & 2 \end{vmatrix}$$

$$\begin{aligned} D_y &= 1 * ((3*2) - (1*10)) - 14 * ((2*2) - (1*3)) + 3 * ((2*10) - (3*3)) \\ &= -4 - 14 + 33 = 15 \end{aligned}$$

$$y = \frac{D_y}{D} = \frac{15}{10} = 1.5$$

$$3- D_z = \begin{vmatrix} 1 & 2 & 14 \\ 2 & -1 & 3 \\ 3 & 1 & 10 \end{vmatrix}$$

$$\begin{aligned} D_z &= 1 * ((-1*10) - (3*1)) - 2 * ((2*10) - (3*3)) + 14 * ((2*1) - (-1*3)) \\ &= -13 - 22 + 70 = 35 \end{aligned}$$

$$z = \frac{D_z}{D} = \frac{35}{10} = 3.5$$

$$x = 0.5, y = 1.5, z = 3.5$$

Home Work

1- solve the linear equation by using Gramer rule

$$x + 2y + 3z = 5$$

$$2x + y + 2z = 7$$

$$2x + 2y + z = 2$$

2- solve the linear equation by using Gramer rule

$$2x + 3y + z = 12$$

$$4x + 6y - z = 9$$

$$x + 2y - 3z = -3$$

3- solve the linear equation by using Gramer rule

$$2x + 3y = 8$$

$$x - 4y = -2$$

4- solve the linear equation by using Gramer rule

$$9x - 3y = 9$$

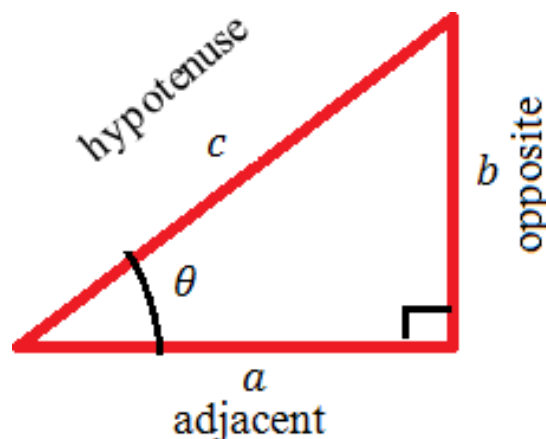
$$3x + y = 3$$

Lecturer 3 : Trigonometric functions and some applications

Trigonometric functions

There are six basic trigonometric functions used in Trigonometry. These functions are trigonometric ratios. The six basic trigonometric functions are sine function, cosine function, secant function, co-secant function, tangent function, and co-tangent function. The trigonometric functions and identities are the ratio of sides of a right-angled triangle. The sides of a right triangle are the perpendicular side, hypotenuse, and base, which are used to calculate the sine, cosine, tangent, secant, cosecant, and cotangent values using trigonometric formulas.

A right triangle is a triangle with a right angle (90°)



For every angle θ in the triangle, there is the side of the triangle adjacent to it, the side opposite of it and the hypotenuse such that $a^2 + b^2 = c^2$.

For angle θ , the trigonometric functions are defined as follows:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hyp}}{\text{adj}} = \frac{c}{a}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hyp}}{\text{opp}} = \frac{c}{b}$$

Trigonometric Functions Values

The trigonometric functions have a domain θ , which is in degrees or radians. Some of the principal values of θ for the different trigonometric functions are presented below in a table.

θ degrees	0°	30°	45°	60°	90°	180°	270°	360°
θ radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
$\csc \theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞	1
$\cot \theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞

Trigonometric functions of negative angles

$$\sin(-\theta) = -\sin \theta \quad , \quad \cos(-\theta) = \cos \theta \quad \text{and} \quad \tan(-\theta) = -\tan \theta$$

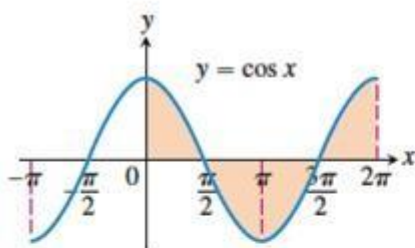
Some useful relationships among trigonometric functions

$$1. \quad \sin^2 x + \cos^2 x = 1, \quad \sec^2 x - \tan^2 x = 1, \quad \csc^2 x - \cot^2 x = 1$$

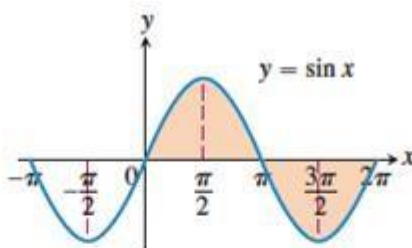
$$2. \quad \sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$3. \quad \sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

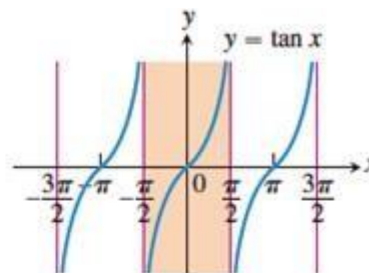
Graphs of Trigonometric Functions



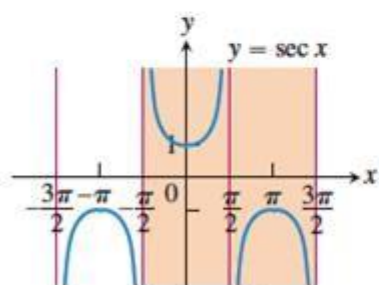
Domain: $-\infty < x < \infty$
 Range: $-1 \leq y \leq 1$
 Period: 2π



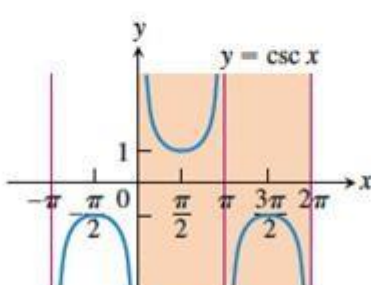
Domain: $-\infty < x < \infty$
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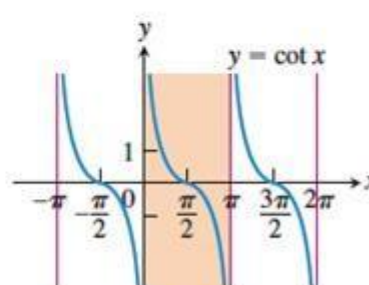
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$
 Range: $-\infty < y < \infty$
 Period: π



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$
 Range: $y \leq -1$ and $y \geq 1$
 Period: 2π



Domain: $x \neq 0, \pm\pi, \pm2\pi, \dots$
 Range: $y \leq -1$ and $y \geq 1$
 Period: 2π



Domain: $x \neq 0, \pm\pi, \pm2\pi, \dots$
 Range: $-\infty < y < \infty$
 Period: π

Derivatives of trigonometric functions

If u is a function x , the chain rule version of this differentiation rule is

1. $\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$
2. $\frac{d}{dx}(\cos u) = -\sin u \cdot \frac{du}{dx}$
3. $\frac{d}{dx}(\tan u) = \sec^2 u \cdot \frac{du}{dx}$
4. $\frac{d}{dx}(\cot u) = -\csc^2 u \cdot \frac{du}{dx}$
5. $\frac{d}{dx}(\sec u) = \sec u \tan u \cdot \frac{du}{dx}$
6. $\frac{d}{dx}(\csc u) = -\csc u \cot u \cdot \frac{du}{dx}$

Example 1: Find derivatives of the functions

1. $y = \sin^2 x \Rightarrow y = (\sin x)^2 \Rightarrow \frac{dy}{dx} = 2 \sin x \cos x = \sin 2x$
2. $y = \cos(x^2) \Rightarrow \frac{dy}{dx} = -2x \sin(x^2)$
3. $y = \tan \sqrt{x} \Rightarrow \frac{dy}{dx} = \sec^2 \sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{\sec^2 \sqrt{x}}{2\sqrt{x}}$
4. $y = x^2 \sec 3x \Rightarrow \frac{dy}{dx} = 3x^2 \sec 3x \tan 3x + 2x \sec 3x$
 $= (x \sec 3x)(2 + 3x \tan 3x)$
5. $y = \sqrt{\sin 2x} \Rightarrow y = (\sin 2x)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2} (\sin 2x)^{-1/2} \times \cos 2x \times 2$
 $= \frac{\cos 2x}{\sqrt{\sin 2x}}$

Example 2: If $y = \tan 2t$ and $x = \sec 2t$ show that $\frac{dy}{dx} = \csc 2t$

$$\begin{aligned} \frac{dy}{dt} &= 2 \sec^2 2t, & \frac{dx}{dt} &= 2 \sec 2t \tan 2t \\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = 2 \sec^2 2t \times \frac{1}{2 \sec 2t \tan 2t} = \frac{\sec 2t}{\tan 2t} \\ &= \frac{1}{\frac{\sin 2t}{\cos 2t}} = \frac{1}{\sin 2t} = \csc 2t \end{aligned}$$

Example 3: If $y = \theta - \cos \theta$ and $x = \theta + \cos \theta$; $(0 \leq \theta \leq \frac{\pi}{2})$ show that $\frac{dy}{dx}$

$$= (\sec \theta + \tan \theta)^2$$

$$\frac{dy}{d\theta} = 1 + \sin \theta \quad \text{and} \quad \frac{dx}{d\theta} = 1 - \sin \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{1 + \sin \theta}{1 - \sin \theta}$$

$$\frac{dy}{dx} = \frac{1 + \sin \theta}{1 - \sin \theta} \times \frac{1 + \sin \theta}{1 + \sin \theta} = \frac{1 + 2 \sin \theta + \sin^2 \theta}{1 - \sin^2 \theta} = \frac{1 + 2 \sin \theta + \sin^2 \theta}{\cos^2 \theta}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 \theta} + \frac{2 \sin \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \sec^2 \theta + 2 \sec \theta \tan \theta + \tan^2 \theta = (\sec \theta + \tan \theta)^2$$

Inverse trigonometric functions

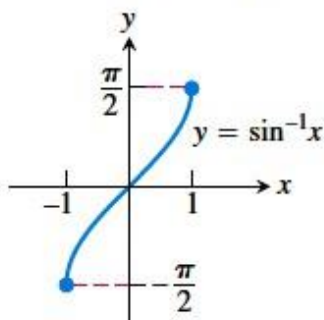
The inverse trigonometric functions are defined to be the inverses of particular parts of the trigonometric functions; parts that do have inverses. The inverse sine function, denoted by $\sin^{-1} x$ (some books use the notation $\arcsin(x)$), is defined to be the inverse of the restricted sine function. A similar idea holds for all the other inverse trigonometric functions. It is important here to note that in this case the “- 1” is not an

exponent and so, $\sin^{-1} x \neq \frac{1}{\sin x}$

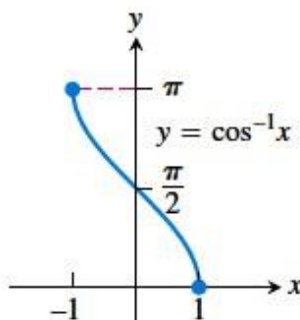
In inverse trigonometric functions the “- 1” looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trigonometric function. It is a notation that we use in this case to denote inverse trigonometric functions. If we had really wanted exponentiation to denote 1 over sine, we would use the following:

$$(\sin x)^{-1} = \frac{1}{\sin x}$$

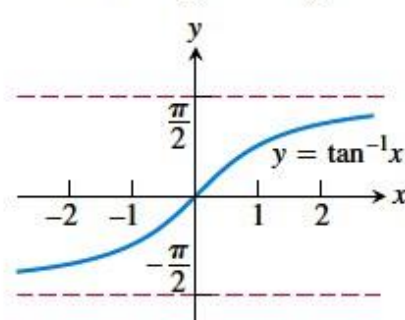
Domain: $-1 \leq x \leq 1$
 Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



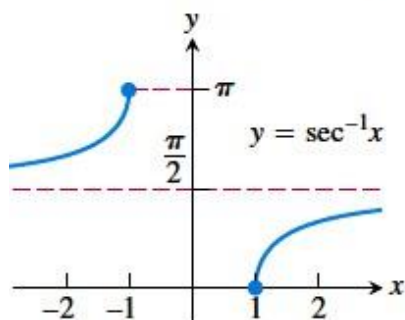
Domain: $-1 \leq x \leq 1$
 Range: $0 \leq y \leq \pi$



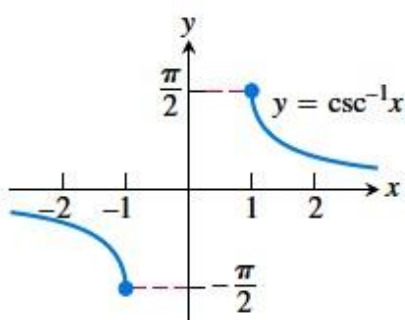
Domain: $-\infty < x < \infty$
 Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



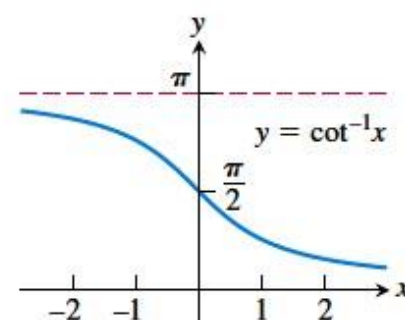
Domain: $x \leq -1$ or $x \geq 1$
 Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$



Domain: $x \leq -1$ or $x \geq 1$
 Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$



Domain: $-\infty < x < \infty$
 Range: $0 < y < \pi$



Derivatives of inverse trigonometric functions

Let u be a function x , the derivatives of inverse trigonometric functions are:

$$\begin{array}{ll}
 1. \frac{d}{dx} (\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} & 2. \frac{d}{dx} (\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \\
 3. \frac{d}{dx} (\tan^{-1} u) = \frac{1}{1+u^2} \cdot \frac{du}{dx} & 4. \frac{d}{dx} (\cot^{-1} u) = \frac{-1}{1+u^2} \cdot \frac{du}{dx} \\
 5. \frac{d}{dx} (\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx} & 6. \frac{d}{dx} (\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}
 \end{array}$$

Example 4: Find the derivative for

$$1. y = \sin^{-1} 2x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \times 2 = \frac{2}{\sqrt{1-4x^2}}$$

$$\begin{aligned}
 2. y &= 3x \cos^{-1} 3x - \sqrt{1-9x^2} \\
 \frac{dy}{dx} &= 3x \times \frac{-1}{\sqrt{1-(3x)^2}} \times 3 + 3 \cos^{-1} 3x - \frac{-18x}{2\sqrt{1-9x^2}} \\
 &= \frac{-9x}{\sqrt{1-9x^2}} + 3 \cos^{-1} 3x + \frac{9x}{\sqrt{1-9x^2}} = 3 \cos^{-1} 3x
 \end{aligned}$$

$$3. y = 2\sqrt{x} \tan^{-1} \sqrt{x}$$

$$\frac{dy}{dx} = 2\sqrt{x} \times \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} + 2 \tan^{-1} \sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{1}{1+x} + \frac{\tan^{-1} \sqrt{x}}{\sqrt{x}}$$

Exercises

Find derivative in each of the following problems(1 – 4)

$$1. y = \sec^2 2x$$

$$2. y = x^2 \sin x + 2x \cos x - 2 \sin x$$

$$3. y = \sqrt{x^2 - 1} - \sec^{-1} x$$

$$4. y = 2x \cos^{-1} \sqrt{x} + \sin^{-1} \sqrt{x} - 2\sqrt{x - x^2}$$

$$5. \text{ If } y = 1 - \sin \theta \text{ and } x = \theta - \sin \theta \text{ find } \frac{dy}{dx}$$

$$6. \text{ If } y = \sec^{-1} t \text{ and } x = \sqrt{t^2 - 1} \text{ find } \frac{dy}{dx}$$

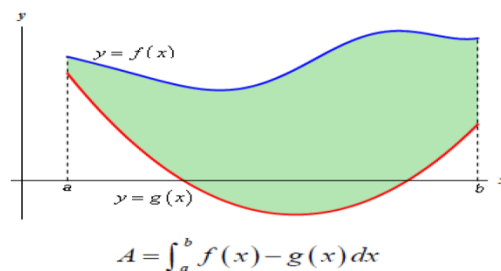
Lecture 11: application of integration

Application of integration

1. Area Between Two Curves

A- Integration with respect to x

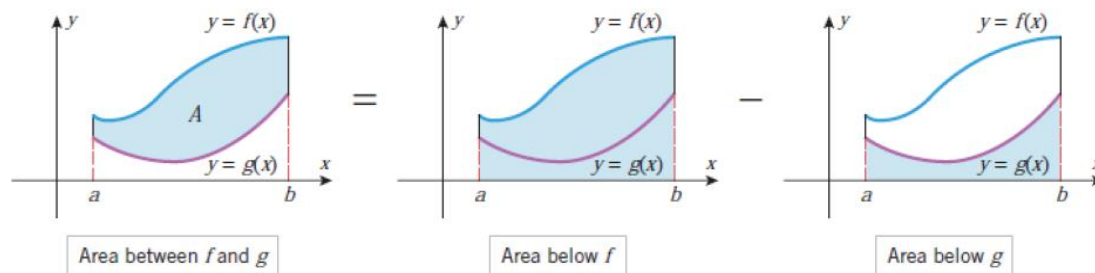
In the first case the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ is determined. If it is assumed that $f(x) \geq g(x)$.



B- Integration with respect to y

The area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$ is given by :

$$A = \int_c^d f(y) - g(y) dy$$



Ex: Find the area bounded by the x -axis and the curve : $y = 2x - x^2$.

Sol: firstly the point of intersection between the curve and x - axis is calculated by making $y = 0$,

$$2x - x^2 = 0$$

$x(2 - x) = 0$. Either $x = 0$ or $x = 2$, $A = \int_0^2 (2x - x^2) dx =$

$$\left[x^2 - \frac{x^3}{3} \right]_0^2 = 4 - \frac{8}{3} = 1.33 \text{ squared unit}$$

Ex: Find the area bounded by the y -axis and the curve $x = y^2 - y^3$

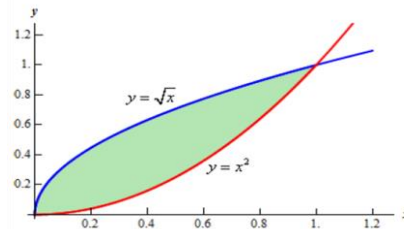
Sol: For $x = 0$ then $y^2(1 - y) = 0$, $y = 0$ and $y = 1$

$$A = \int_0^1 (y^2 - y^3) dy = \left[\frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Ex: Determine the area of the region enclosed by: $y = x^2$ and $y = \sqrt{x}$

Sol: For $y = \sqrt{x} - x^2 = 0$, $\sqrt{x}(1 - x^{3/2}) = 0$, $x = 0$ and $x = 1$

$$A = \int_0^1 (\sqrt{x} - x^2) dx = \left[\frac{2x^{3/2}}{3} - \frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$



Ex: Determine the area of the region bounded by: $y = 2x^2 + 10$ and $y = 4x + 16$

Sol: For $y = 4x + 16 - 2x^2 - 10 = -2x^2 + 4x + 6$

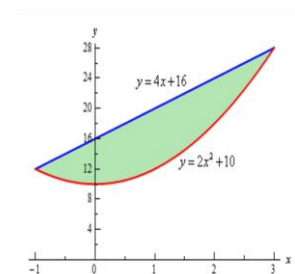
$$y = 0, -2x^2 + 4x + 6 = 0$$

$$x^2 - 2x - 3 = (x - 3)(x + 1)$$

$$x = 3 \text{ and } x = -1$$

$$A = \int_{-1}^3 [-2x^2 + 4x + 6] dx = \left[-\frac{2x^3}{3} + 2x^2 + 6x \right]_{-1}^3$$

$$= \left[\frac{-2 * 27}{3} + 2 * 9 + 6 * 3 - \left(\frac{+2}{3} + 2 - 6 \right) \right]$$

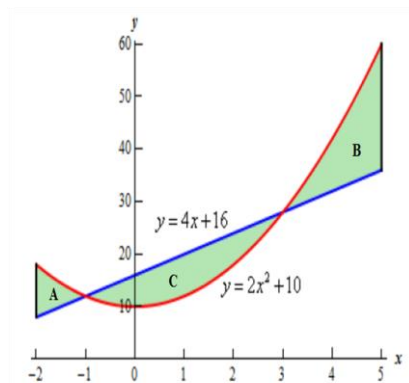


$$= -18 + 18 + 18 - (-3.334) = 21.334$$

Ex: Determine the area of the region bounded by: $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$, and $x = 5$

Sol: The area from $x = -1$ to $x = 3$ is (area C) calculated from the above example. Areas (A&B) are calculated by:

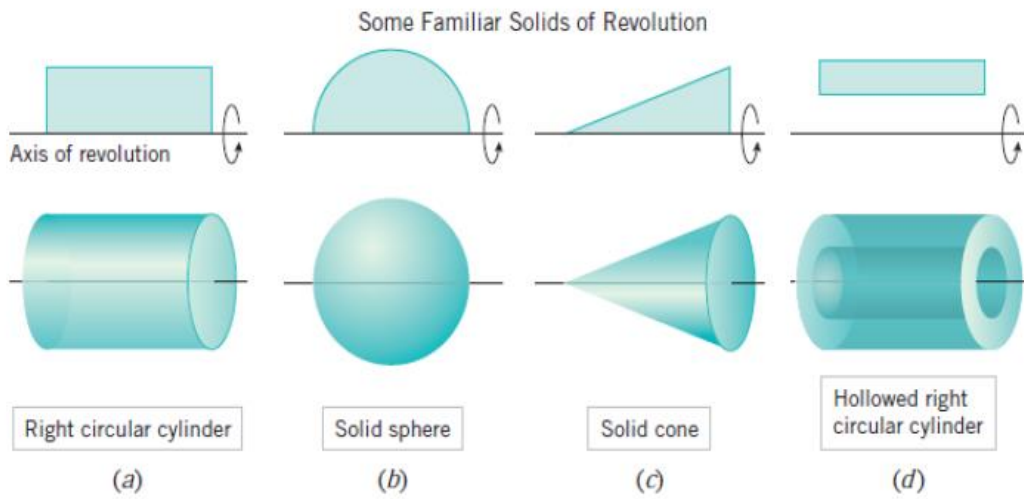
$$\begin{aligned}
 A_{A\&B} &= \int_{-2}^{-1} (2x^2 + 10 - 4x - 16)dx + \\
 &\int_3^5 (2x^2 + 10 - 4x - 16)dx \\
 &= \int_{-2}^{-1} (2x^2 - 4x - 6)dx + \int_3^5 (2x^2 - 4x - 6)dx \\
 &= \left[\frac{2x^3}{3} - 2x^2 - 6x \right]_{-2}^{-1} + \left[\frac{2x^3}{3} - 2x^2 - 6x \right]_3^5 \\
 &= \frac{2(-1+8)}{3} - 2(1-4) - 6(-1+2) + 2\frac{125-27}{3} - 2(25-9) \\
 &\quad - 6(5-3) \\
 &= \frac{14}{3} + 6 - 6 + 2\frac{98}{3} - 2(16) - 12 = \frac{14}{3} + \frac{196}{3} - 32 - 12 = \frac{210}{3} - 44 \\
 &= 70 - 44 = 26
 \end{aligned}$$



The total area is $A_{A\&B} + A_C = 26 + 21.3 = 47.3$

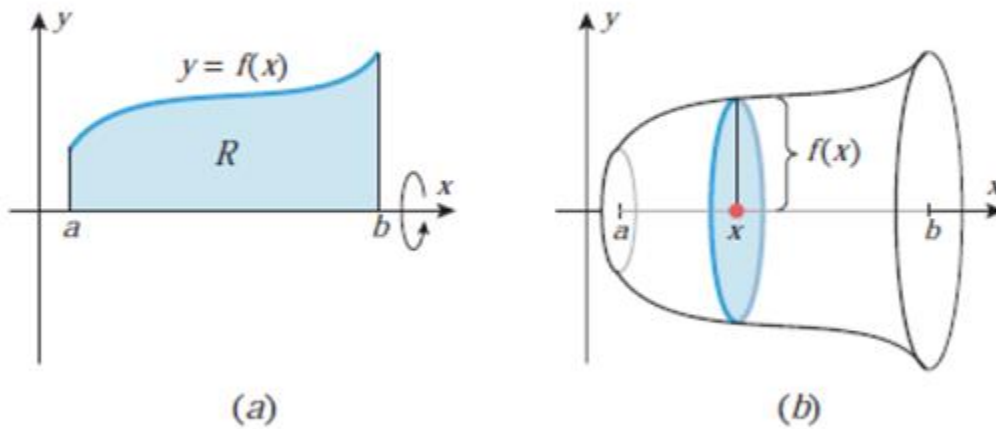
2. Volume of the Solids of revolution

A Solid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the **axis of revolution**.



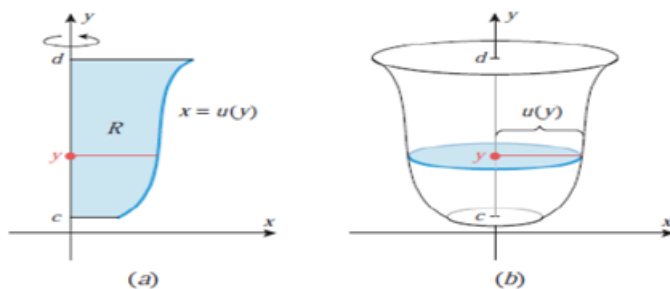
The Disk method

A) Revolution about x- axis



Revolution about x-axis

B) Revolution about Y- axis



Disks

Revolution about Y-axis

$V = \int_a^b A(x) dx$	$V = \int_c^d A(y) dy$
------------------------	------------------------

$$A = \pi(\text{radius})^2$$

$$A = \pi(R_{(x)})^2$$

$$A = \pi(R_{(y)})^2$$

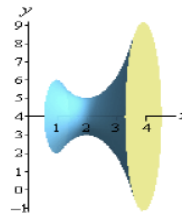
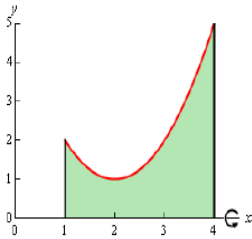
$$A(x) = \pi[f(x)]^2$$

$$V = \int_a^b \pi[f(x)]^2 dx$$

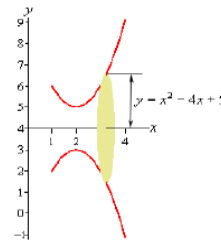
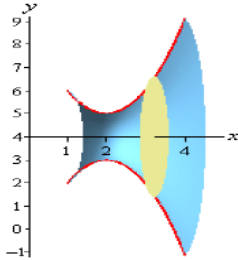
$$V = \int_c^d \pi[u(y)]^2 dy$$

Ex: Determine the volume of the solid generated by rotating the region bounded by $f(x) = x^2 - 4x + 5$ and $x = 1, x = 4$ about the x-axis.

Step 1 is to sketch the bounding region and the solid obtained by rotating the region about the x -axis. Here are both of these sketches.



Step 2: To get a cross section we cut the solid at any x , since the x -axis is the axis of rotation.



$$A(x) = \pi r^2 = \pi [f(x)]^2$$

$$A(x) = \pi(x^2 - 4x + 5)^2 = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25)$$

Step 3. Determine the boundaries which will represent the limits of integration. Working from left to right the first cross section will occur at $x = 1$, and the last cross section will occur at $x = 4$. These are the limits of integration.

Step 4. Integrate to find the volume:

$$V = \int_a^b A(x) dx = \pi \int_a^b f(x) dx = \pi \int_1^4 (x^2 - 4x + 5)^2 dx = \pi \int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25) dx = \pi \left(\frac{1}{5} x^5 - 2x^4 + \frac{26}{3} x^3 - 20x^2 + 25x \right) \Big|_1^4 = \frac{78\pi}{5}$$

2. Finding volume of a solid of revolution using a washer method.

This is an extension of the disc method. The procedure is essentially the same, but now we are dealing with a hollowed object and two functions instead of one, so we have to take the difference of these functions into the account.

The general formula in this case would be:

$$A = \pi(R^2 - r^2) \text{ where } R \text{ is an outer radius and } r \text{ is the inner radius.}$$

FORMULAS: $V = \int A(x)dx$, or respectively $\int A(y)dy$

1. The volume of the solid generated by a region between $f(x)$ and $g(x)$ bounded by the vertical lines $x=a$ and $x=b$, which is revolved about the x -axis is

$$V = \pi \int_a^b \left| (f(x))^2 - (g(x))^2 \right| dx \quad (\text{washer with respect to } x)$$

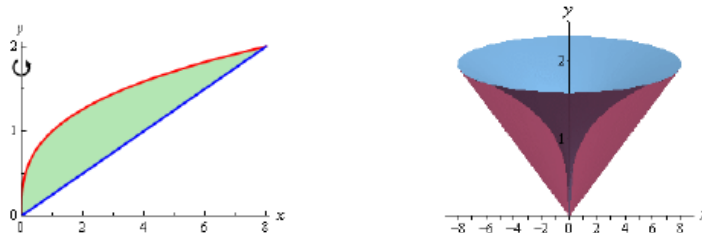
2. The volume of the solid generated by a region between $f(y)$ and $g(y)$ bounded by the horizontal lines $y=c$ and $y=d$ which is revolved about the y -axis.

$$V = \pi \int_c^d \left| (f(y))^2 - (g(y))^2 \right| dy \quad (\text{washer with respect to } y)$$

Ex: Determine the volume of the solid generated by rotating the region bounded by $y = \sqrt[3]{x}$, and $y = \frac{x}{4}$ that lies in the first quadrant about the y -axis.

Solution

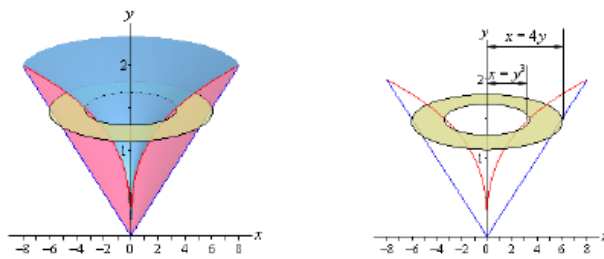
Step 1: Graph the bounding region and a graph of the object. The cross section is cut perpendicular to the axis of rotation and it is a horizontal washer. The inner and outer radii of the washer are x values, so we will need to rewrite our functions into the form $x = f(y)$.



Here are the functions written in the correct form for this example.

$$y = \sqrt[3]{x} \Rightarrow x = y^3 \quad \text{and} \quad y = \frac{x}{4} \Rightarrow x = 4y$$

Step 2. Graph couple of sketches of the boundaries of the walls of this object as well as a typical washer. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



The cross-sectional area is then, $A(y) = \pi((4y)^2 - (y^3)^2) = \pi(16y^2 - y^6)$

Step 3. Working from the bottom of the solid to the top we can see that the first cross-section will occur at $y=0$ and the last cross-section will occur at $y=2$. These will be the limits of integration.

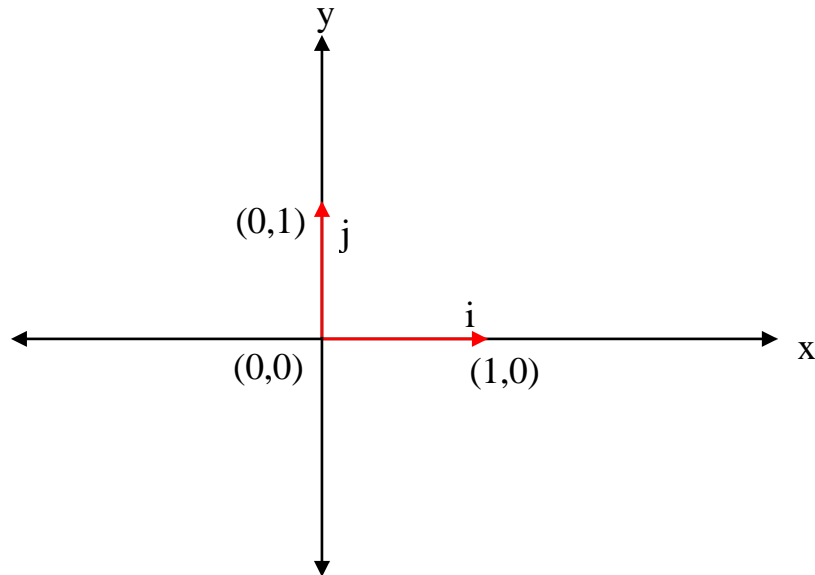
Step 4. The volume is then, $V = \int_c^d A(y)dy = \pi \int_0^2 (16y^2 - y^6)dy = \pi \left(\frac{16}{3}y^3 - \frac{1}{7}y^7 \right) \Big|_0^2 = \frac{512\pi}{21}$

Lecture 3 : Vectors**Vector components:**

Quantities can be divided into:

- 1) scalar: which is defined by magnitude only, for example mass, time...
- 2) vector: the quantity which needs magnitude and direction, for example force, velocity, acceleration....

We shall denote the vector from $(0,0)$ to $(1,0)$ by (i) and the vector from $(0,0)$ to $(0,1)$ by (j) as shown in figure below:



Then any vector in the $x - y$ plane can be divided in the terms of i and j .

Ex. Sketch the following vectors:

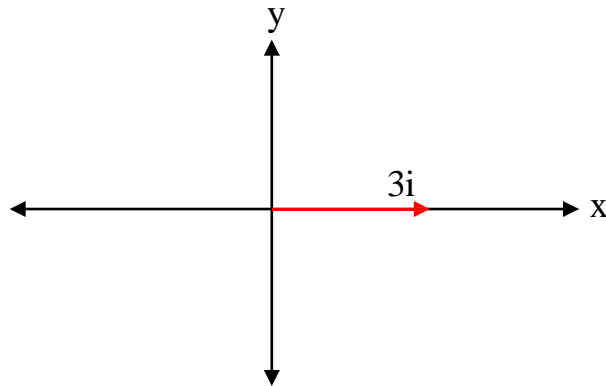
a) $3i$

b) $-4i$

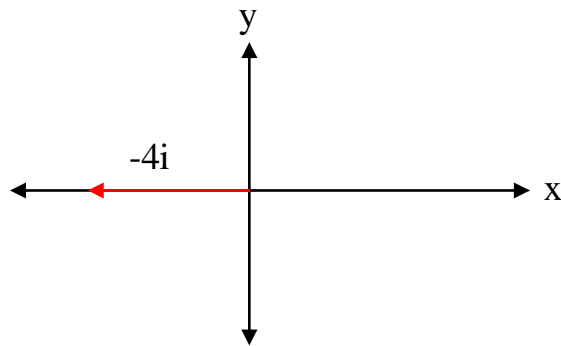
c) $-2i + 3j$

Sol.:

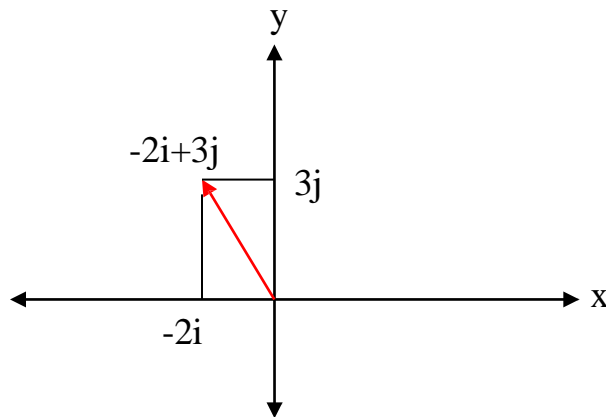
a)



b)



c)



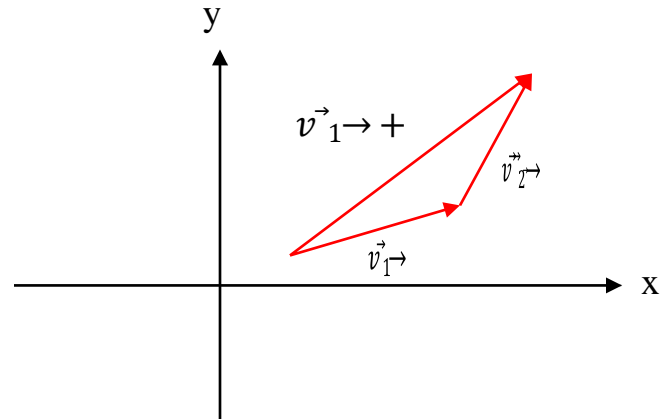
Note: when we say the vector $A \rightarrow \rightarrow B \rightarrow$ means: This vector directed from point A to point B.

Arithmetic operation on vector:

1) Addition:

Let $v_{1 \rightarrow} = a_1i + b_1j$ and $v_{2 \rightarrow} = a_2i + b_2j$

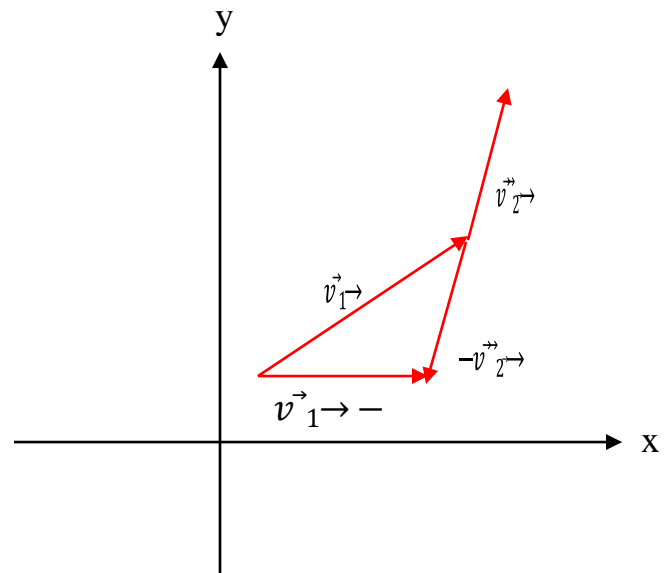
$$\begin{aligned} \text{Then } v_{1 \rightarrow} + v_{2 \rightarrow} &= a_1i + b_1j + a_2i + b_2j \\ &= (a_1 + a_2)i + (b_1 + b_2)j \end{aligned}$$



2) Subtraction:

Let $v_{1 \rightarrow} = a_1i + b_1j$ and $v_{2 \rightarrow} = a_2i + b_2j$

$$\begin{aligned} \text{Then } v_{1 \rightarrow} - v_{2 \rightarrow} &= (a_1i + b_1j) - (a_2i + b_2j) \\ &= (a_1 - a_2)i + (b_1 - b_2)j \end{aligned}$$



3) Multiplication of vector by scalar:

Let $v = ai + bj$ and c is scalar.

$$\text{then } cv = c(ai + bj) = cai + cbj$$

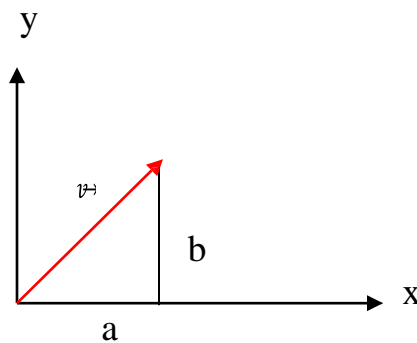
Definitions:**1) Length of vector:**

The length of vector $\vec{v} = ai + bj$ is usually denoted by $|\vec{v}|$ which may be read as "The magnitude of \vec{v} ".

$|\vec{v}|$ is found using Pythagoras theorem and as follows:

$$\vec{v} = ai + bj$$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$

**2) Zero vector:**

Any vector of length zero is called zero vector $\vec{0}$,

$$ai + bj = 0$$

$$ai + bj = 0i + 0j \text{ if and only if } a = b = 0.$$

3) Unit vector:

It is part from any vector. This part has length equal to unity and it is used to describe the direction of the vector.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} \quad \text{where } \vec{u} \text{ is unit vector of } \vec{v}.$$

Ex. Find the unit vector of $\vec{v} = ai + bj$ and prove that unit vector depends on the angle between the vector and the x - axis?

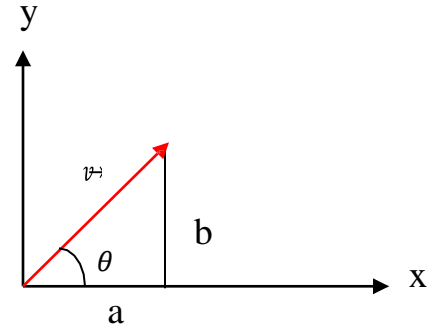
Sol.:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{u} = \frac{ai + bj}{\sqrt{a^2 + b^2}}$$

$$\vec{u} = \frac{a}{\sqrt{a^2 + b^2}}i + \frac{b}{\sqrt{a^2 + b^2}}j$$

$$\vec{u} = \cos\theta i + \sin\theta j$$



Ex. Find the unit vector of $\vec{A} = 3i + 4j$.

Sol.:

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|}$$

$$|\vec{A}| = \sqrt{3^2 + 4^2} = 5$$

$$\vec{u} = \frac{3i+4j}{5} = 0.6i + 0.8j$$

since

$$\vec{u} = \cos\theta i + \sin\theta j$$

$$\cos\theta = 0.6$$

$$\therefore \theta = 53^\circ$$

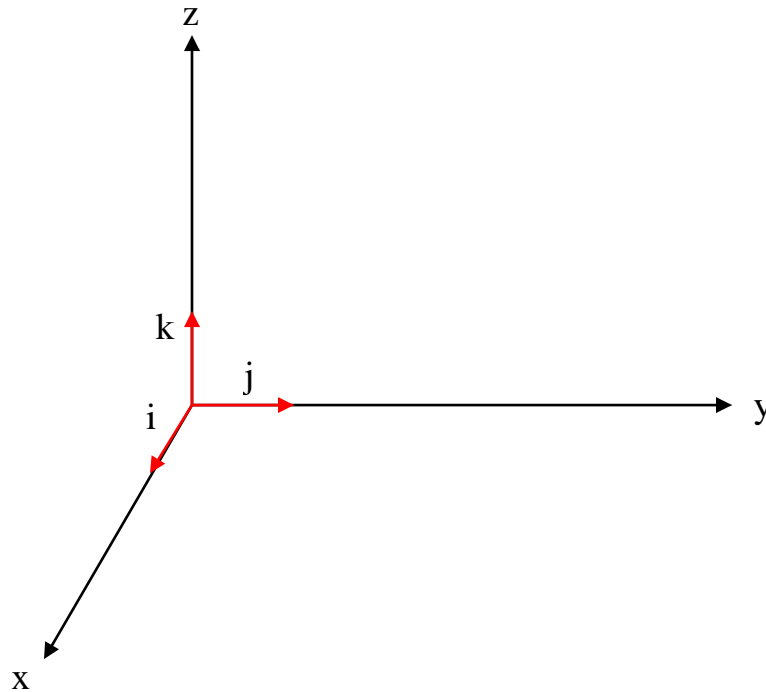
Vector in Space:

Now we shall consider the vector in three dimensional space as follows:

(i) as a vector pointing from (0,0,0) to (1,0,0)

(j) as a vector pointing from (0,0,0) to (0,1,0)

(k) as a vector pointing from (0,0,0) to (0,0,1)



Any vector $A \rightarrow$ for example may be represented as:

$$A \rightarrow = ai + bj + ck$$

and

$$|A \rightarrow| = \sqrt{a^2 + b^2 + c^2}$$

Ex.: Find a unit vector in the direction of vector from $p_1(1,0,1)$ and $p_2(3,2,0)$.

Sol.:

$$\vec{p}_1 \rightarrow \vec{p}_2 = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k}$$

$$\vec{p}_1 \rightarrow \vec{p}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\vec{p}_1 \rightarrow \vec{p}_2| = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\vec{u} = \frac{\vec{p}_1 \rightarrow \vec{p}_2}{|\vec{p}_1 \rightarrow \vec{p}_2|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

Scalar product (dot product):

The scalar or dot product of two vectors $A \rightarrow$ and $B \rightarrow$, denoted by $A \rightarrow \cdot B \rightarrow$ (read $A \rightarrow$ dot $B \rightarrow$), is defined as the product of the magnitudes of $A \rightarrow$ and $B \rightarrow$ and the cosine of the angle θ between them.

$$A \rightarrow \cdot B \rightarrow = |A \rightarrow| \cdot |B \rightarrow| \cos\theta, \quad 0 \leq \theta \leq \pi$$

The following laws was valid:

1. $A \rightarrow \cdot B \rightarrow = B \rightarrow \cdot A \rightarrow$
2. $A \rightarrow \cdot (B \rightarrow + C \rightarrow) = A \rightarrow \cdot B \rightarrow + A \rightarrow \cdot C \rightarrow$
3. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

The dot product can be used to find:

- 1) The angle between two vectors.
- 2) The projection of vector $B \rightarrow$ on $A \rightarrow = |B \rightarrow| \cos\theta = \frac{A \rightarrow \cdot B \rightarrow}{|A \rightarrow|}$
- 3) The projection of $A \rightarrow$ on $B \rightarrow = |A \rightarrow| \cos\theta = \frac{A \rightarrow \cdot B \rightarrow}{|B \rightarrow|}$

Ex. Find the angle between $\vec{A} = i - 2j - 2k$ and $\vec{B} = 6i + 3j + 2k$ also find the projection of \vec{A} on \vec{B} and \vec{B} on \vec{A} .

Sol.:

$$1) \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos\theta$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$\vec{A} \cdot \vec{B} = 1*6 + (-2)*3 + (-2)*2 = -4$$

$$|\vec{A}| = \sqrt{1 + 4 + 4} = 3$$

$$|\vec{B}| = \sqrt{36 + 9 + 4} = 7$$

$$\cos\theta = \frac{-4}{3*7} = \frac{-4}{21}$$

$$\theta = 101^\circ$$

The projection of $A \rightarrow$ on $B \rightarrow = |A \rightarrow| \cos \theta = \frac{A \rightarrow \cdot B \rightarrow}{|B|} = \frac{-4}{7}$

2) The projection of vector $B \rightarrow$ on $A \rightarrow = |B \rightarrow| \cos \theta = \frac{A \rightarrow \cdot B \rightarrow}{|A|} = \frac{-4}{3}$

Orthogonal vectors:

The two vectors $A \rightarrow$ and $B \rightarrow$ are orthogonal if and only if:

$$A \rightarrow \cdot B \rightarrow = 0$$

Lecture 6 : partial derivation

Second order partial derivatives:

Again, let $z = f(x,y)$ be a function of x and y .

- $\frac{\partial^2 z}{\partial x^2}$ means the second derivative with respect to x holding y constant.
- $\frac{\partial^2 z}{\partial y^2}$ means the second derivative with respect to y holding x constant.
- $\frac{\partial^2 z}{\partial x \partial y}$ means differentiate first with respect to y and then with respect to x .

The "mixed" partial derivative $\frac{\partial^2 z}{\partial x \partial y}$ is as important in applications as the others.

It is a general result that: $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

i.e. you get the same answer whichever order the differentiation is done.

Ex: Find all derivative of the following function $y = 3x^3 - 4x^2 + 7x + 10$

Sol:

$$\frac{dy}{dx} = 9x^2 - 8x + 7$$

$$\frac{d^2y}{dx^2} = 18x - 8$$

$$\frac{d^3y}{dx^3} = 18, \frac{d^4y}{dx^4} = 0$$

Ex.: a. Let $z = 4x^2 - 8xy^4 + 7y^5 - 3$. Find all the first and second order partial derivatives of z .

b. Find all the first and second order partial derivatives of the function $z = \sin(xy)$.

Sol.:

a. $z = 4x^2 - 8xy^4 + 7y^5 - 3$

First partial derivative: $\frac{\partial z}{\partial x} = 8x - 8y^4$, $\frac{\partial z}{\partial y} = -32xy^3 + 35y^4$

Second partial derivative: $\frac{\partial^2 z}{\partial x^2} = 8$, $\frac{\partial^2 z}{\partial y^2} = -96xy^2 + 140y^3$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (-32xy^3 + 35y^4) = -32y^3$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (8x - 8y^4) = -32y^3$$

b. $z = \sin(xy)$

First partial derivative: $\frac{\partial z}{\partial x} = y \cos(xy)$, $\frac{\partial z}{\partial y} = x \cos(xy)$

Second partial derivative: $\frac{\partial^2 z}{\partial x^2} = -y^2 \sin(xy)$, $\frac{\partial^2 z}{\partial y^2} = -x^2 \sin(xy)$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (x \cos(xy)) = -xy \sin(xy) + \cos(xy)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (y \cos(xy)) = y(-x \sin(xy)) + \cos(xy) = -xy \sin(xy) + \cos(xy)$$

Subscript notation for second order partial derivatives:

If $z = f(x, y)$ then:

- z_{xx} means $\frac{\partial^2 z}{\partial x^2}$
- z_{yy} means $\frac{\partial^2 z}{\partial y^2}$
- z_{xy} means $\frac{\partial^2 z}{\partial x \partial y}$ or $\frac{\partial^2 z}{\partial y \partial x}$

The chain rule: If $y = f(u)$, $u = g(x)$ and the derivatives $\frac{dy}{du}$ and $\frac{du}{dx}$ both exist then the composite function defined by $f(g(x))$ has a derivative given by:

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$$

Ex: let $y = \frac{1}{t}$, $t = \frac{1}{x} + x^2$ find $\frac{dy}{dx}$

Sol: $\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$

$$\frac{dy}{dx} = -\frac{1}{t^2} * \left(-\frac{1}{x^2} + 2x\right)$$

$$\frac{dy}{dx} = \frac{-1}{\left(\frac{1}{x} + x^2\right)^2} * \left(-\frac{1}{x^2} + 2x\right) = \frac{1 - 2x^3}{(1 + x^3)^2}$$

Ex: Find $\frac{dy}{dx}$, if $y = t^2 - 1$ and $x = 2t + 3$

Sol: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t = \frac{x-3}{2}$

Ex: Find d^2y/dx^2 for $x = t - t^2$ and $y = t - t^3$.

$$\text{Sol: } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{1-2t}$$

$$\begin{aligned} \left(\frac{d^2y}{dx^2}\right) &= \frac{\frac{d^2y}{dt^2}}{\frac{dx}{dt}} = \frac{\frac{(1-2t) * (-6t) - (1-3t^2) * (-2)}{(1-2t)^2}}{1-2t} \\ &= \frac{-6t+12t^2+2-6t^2}{(1-2t)^3} = \frac{6t^2-6t+2}{(1-2t)^3} \end{aligned}$$

Lecture 8 : Integration

Introduction to Integration

Integration is a way of adding slices to find the whole. Integration can be used to find areas, volumes, central points and many useful things.

Basic rules of integration

$$1 - \int a. dx = ax + C$$

$$2 - \int a. x. dx = a \int x. dx = a \frac{x^2}{2} + C$$

$$3 - \int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

$$4 - \int x^n. dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$5 - \int u^n. du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)(u = f(x))$$

Example: find the following integrations:

$$1 - \int 5x^3. dx = \frac{5x^4}{4} + C$$

$$2 - \int \frac{7}{x^6}. dx = \int 7x^{-6}. dx = -\frac{7x^{-5}}{5} + C$$

$$3 - \int \left(4x + \frac{8}{\sqrt[3]{x^2}} \right) dx = \int 4x dx + \int \frac{8dx}{x^{\frac{2}{3}}} = \int 4x dx + \int 8x^{-\frac{2}{3}} dx$$

$$= 2x^2 + 24x^{\frac{1}{3}} + C$$

$$4 - \int x \cdot (3 - x^2)^3 \cdot dx$$

الحل:- لإيجاد تكامل هذه الدالة $(x \cdot (3 - x^2)^3 \cdot dx)$ يجب ان نقسمها الى قسمين احدهما الدالة المراد تكاملها $((3 - x^2)^3)$ بعد استيفاء شروط التكامل والثاني مشتقة $(x \cdot dx)$ تلك الدالة.

نفرض

$$u = 3 - x^2 \rightarrow \rightarrow \rightarrow du = -2x dx$$

لكن المتوفر في الحقيقة من مشتقة الدالة فقط $(x \cdot dx)$ والمطلوب مشتقة داخل القوس $(-2x dx)$. ولتكون عملية التكامل صحيحة يجب توفير كل شروط التكامل وهو الضرب في العدد -2 والقسمة عليها وكما يلي.

$$\int \frac{-2}{-2} x \cdot (3 - x^2)^3 \cdot dx = \frac{-1}{2} \int -2x \cdot (3 - x^2)^3 \cdot dx$$

في الحالة الأخيرة توفرت شروط التكامل

$$\frac{-1}{2} \int -2x \cdot (3 - x^2)^3 \cdot dx = \frac{-1}{2} \left[\frac{(3 - x^2)^4}{4} \right] + C = \frac{-(3 - x^2)^4}{8} + C$$

$$5 - \int \frac{x^2 \cdot dx}{\sqrt[4]{(x^3 + 7)^3}}$$

الحل: أولاً نرتب المعادلة برفع المقام الى البسط ورفع الجذر الى أس

$$\int \frac{x^2 \cdot dx}{\sqrt[4]{(x^3 + 7)^3}} = \int x^2 \cdot (x^3 + 7)^{-\frac{3}{4}} dx$$

$$\text{let } u = x^3 + 7 \rightarrow \rightarrow \rightarrow du = 3x^2 dx ,$$

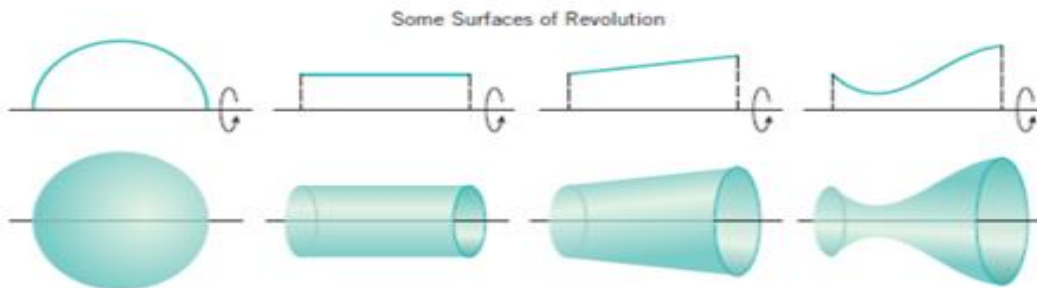
$$\int x^2 \cdot (x^3 + 7)^{-\frac{3}{4}} dx = \frac{1}{3} \int 3x^2 \cdot (x^3 + 7)^{-\frac{3}{4}} dx$$

$$= \frac{1}{3} \left[\frac{(x^3 + 7)^{\frac{1}{4}}}{\frac{1}{4}} \right] + C = \frac{4}{3} (x^3 + 7)^{\frac{1}{4}} + C$$

Lect. 12

Areas of surface of revolution

A surface of revolution is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve.



A) The area of the surface (S) generated by revolving the curve $y = f(x)$ between $x = a$ and $x = b$ about the x -axis.

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

B) The area of the surface (S) generated by revolving the curve $x = g(y)$ between $y = c$ and $y = d$ about the y -axis.

$$S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Example(1)

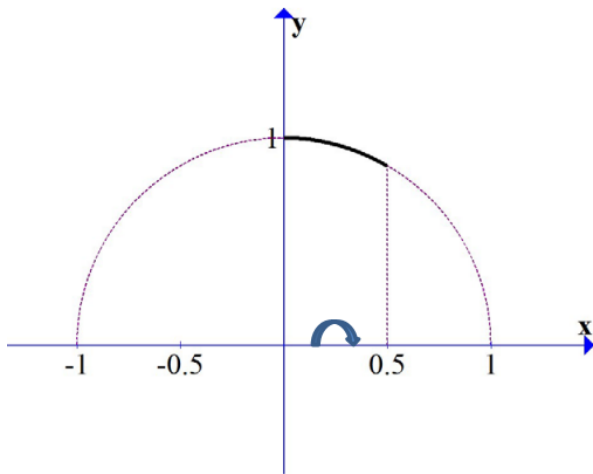
Find the surface area generated by revolving the curve

$$y = \sqrt{1 - x^2}, \quad 0 \leq x \leq \frac{1}{2}$$

about the x -axis.

Solution:

The graph of the curve is the upper semi-circle of radius 1 centered at the origin.



$$\begin{aligned} y &= \sqrt{1 - x^2} \\ \frac{dy}{dx} &= \frac{-x}{\sqrt{1 - x^2}} \end{aligned}$$

$$\begin{aligned} S &= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx \\ &= \int_0^{1/2} 2\pi \sqrt{1 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2} dx \\ &= \int_0^{1/2} 2\pi \sqrt{1 - x^2} \sqrt{1 + \frac{x^2}{1 - x^2}} dx \\ &= \int_0^{1/2} 2\pi \sqrt{1 - x^2} \sqrt{\frac{1}{1 - x^2}} dx \\ &= \int_0^{1/2} 2\pi dx \\ &= \dots = \pi \end{aligned}$$

Find the surface area generated by revolving the curve

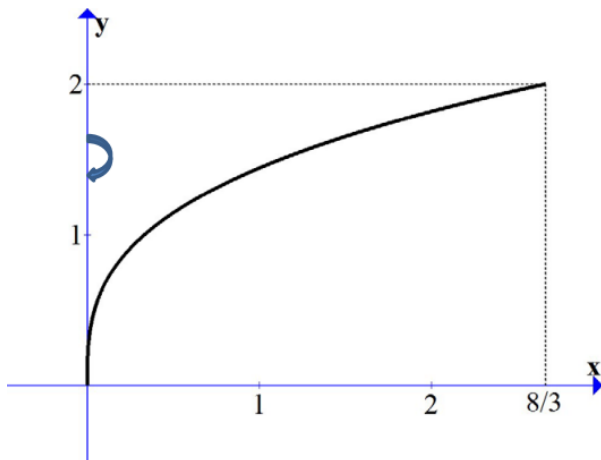
$$y = \sqrt[3]{3x}, \quad 0 \leq y \leq 2$$

about the y -axis.

Solution:

$$y = \sqrt[3]{3x} \Rightarrow x = \frac{1}{3}y^3$$

$$\frac{dx}{dy} = y^2$$



$$\begin{aligned} S &= \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy \\ &= \int_0^2 2\pi \left(\frac{1}{3}y^3\right) \sqrt{1 + (y^2)^2} dy \\ &= \int_0^2 \frac{2\pi}{3} y^3 \sqrt{1 + y^4} dy \end{aligned}$$

$$\begin{aligned} u &= 1 + y^4 \\ du &= 4y^3 dy \Rightarrow \frac{1}{4} du = y^3 dy \\ y = 2 &\Rightarrow u = 17 \\ y = 0 &\Rightarrow u = 1 \end{aligned}$$

$$\begin{aligned} S &= \int_0^2 \frac{2\pi}{3} y^3 \sqrt{1+y^4} dy \\ &= \int_1^{17} \frac{2\pi}{3} \sqrt{u} \cdot \frac{1}{4} du \\ &= \dots = \frac{\pi}{9} (17\sqrt{17} - 1) \end{aligned}$$

Lecture 4 : Derivative

The process of calculating a derivative is called **differentiation**. The notations:

$$f'(x), y', dy/dx, \frac{d}{dx}[f(x)]$$

Derivative by definition:

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Ex: Find the derivative of the function $f(x) = \frac{1}{\sqrt{x}}$

Sol:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{\sqrt{x + \Delta x}} - \frac{1}{\sqrt{x}}}{\Delta x} = \frac{0}{0} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x} - \sqrt{x + \Delta x}}{\Delta x \sqrt{x} \sqrt{x + \Delta x}} * \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x - x - \Delta x}{\Delta x (\sqrt{x + \Delta x})(\sqrt{x} + \sqrt{x + \Delta x})} = \frac{-1}{x(2\sqrt{x})} \end{aligned}$$

Ex: Find the derivative of the function $f(x) = x^2$

Sol:

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = 2x$$

Rules of derivatives:

Let c and n are constants u, v, and w are differentiable functions of x.

1. $\frac{d}{dx} c = 0$
2. $\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx} \Rightarrow \frac{d}{dx} \left(\frac{1}{u} \right) = -\frac{1}{u^2} \frac{du}{dx}$
3. $\frac{d}{dx} c u = c \frac{du}{dx}$
4. $\frac{d}{dx} (u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx} ; \frac{d}{dx} (u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$
5. $\frac{d}{dx} (u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$, and
 $\frac{d}{dx} (u \cdot v \cdot w) = u \cdot v \frac{dw}{dx} + u \cdot w \frac{dv}{dx} + v \cdot w \frac{du}{dx}$
6. $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ where $v \neq 0$

Ex: Find dy/dx for the following functions

1. $y = (x^2 + 2x + 1)^5$

$$\frac{dy}{dx} = 5(x^2 + 2x + 1)^4 * (2x + 2)$$

2- $y = \frac{x^2-1}{x^2+x-2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2 + x - 2) * (2x) - (x^2 - 1)(2x + 1)}{(x^2 + x - 2)^2} \\ &= \frac{2x^3 + 2x^2 - 4x - 2x^3 - x^2 + 2x + 1}{(x^2 + x - 2)^2} = \frac{x^2 - 2x + 1}{(x^2 + x - 2)^2} \end{aligned}$$

3. $y = e^{(x+e^{5x})}$

$$\frac{dy}{dx} = e^{(x+e^{5x})} (1 + 5e^{5x})$$

4. $y = (2^x)^2$

$$\frac{dy}{dx} = 2 * 2^x * 2^x \ln 2 = 2^{x+1} \ln 2$$

$$5. \quad y = e^{\sqrt{1+x^2}}$$
$$\frac{dy}{dx} = e^{\sqrt{1+x^2}} * \frac{2x}{2\sqrt{1+x^2}}$$

$$6. \quad y = 2^x * 3^x$$
$$\frac{dy}{dx} = 2^x \cdot 3^x \ln 3 + 3^x * 2^x \ln 2$$
$$\frac{dy}{dx} = 2^x \cdot 3^x (\ln 3 + \ln 2)$$

Home work:

Using the definition of derivative , find the derivative of the following functions:

$$y = \frac{1}{x+1} \quad y = \frac{x^2}{(x-1)^2} \quad y = \sqrt[3]{x} \quad y = \frac{x^2}{(x-1)^2}$$

Lecture 7 : Limits and Continuity

Definition of Limits: Let f be a function defined on an open interval containing c (possibly except at c) and let L be a real number, then $\lim_{x \rightarrow c} f(x) = L$, which is read as "the limit of $f(x)$ as x approaches c is L "

Limits properties

If L , M , and k , are real numbers and

$$\lim_{x \rightarrow \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \rightarrow \pm\infty} (f(x) + g(x)) = L + M$

2. *Difference Rule:* $\lim_{x \rightarrow \pm\infty} (f(x) - g(x)) = L - M$

3. *Product Rule:* $\lim_{x \rightarrow \pm\infty} (f(x) \cdot g(x)) = L \cdot M$

4. *Constant Multiple Rule:* $\lim_{x \rightarrow \pm\infty} (k \cdot f(x)) = k \cdot L$

5. *Quotient Rule:* $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \rightarrow \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

7. If $f(x) = k$, then $\lim_{x \rightarrow c} f(x) = k$ where c and k are real numbers.

8. Polynomial $\lim_{x \rightarrow c} (a_0 + a_1x + a_2x^2 + \dots + a_nx^n) = a_0 + a_1 * c + a_2 * c^2 + \dots + a_n * c^n$.

9.

The following facts are some times abbreviated by saying :

- a) As x approaches 0 from the right, $1/x$ tends to ∞ .
- b) As x approaches 0 from the left, $1/x$ tends to $-\infty$.
- c) As x tends to ∞ , $1/x$ approaches 0 .
- d) As x tends to $-\infty$, $1/x$ approaches 0 .

Notes:

1. $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 * \infty$ are all undefined quantities.
2. If any constant is divided by ∞ the result is zero.

Ex1: Find the following limits:

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) &= \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} \\ &= 5 + 0 = 5 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{\pi\sqrt{3}}{x^2} &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x} \\ &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} \\ &= \pi\sqrt{3} \cdot 0 \cdot 0 = 0 \end{aligned}$$

(c) Numerator and Denominator of Same Degree

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} \quad \text{Divide numerator and denominator by } x^2. \\ &= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} \end{aligned}$$

(d) Degree of Numerator Less Than Degree of Denominator

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow -\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)} \\ &= \frac{0 + 0}{2 - 0} = 0 \end{aligned}$$

Note: In rational functions divide both the numerator and denominator by the largest power of x in the denominator as:

1. $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = 0$, If $\text{deg.}(f) < \text{deg.}(g)$.
2. $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = \text{cons.}$ If $\text{deg.}(f) = \text{deg.}(g)$
3. $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = \infty$. If $\text{deg.}(f) > \text{deg.}(g)$

Ex2: Find the following limits:

- 1) $\lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2}$, 2) $\lim_{x \rightarrow a} \frac{x^3 - a^3}{x^4 - a^4}$
 3) $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 3x}$, 4) $\lim_{y \rightarrow 0} \frac{\tan 2y}{3y}$
 5) $\lim_{x \rightarrow 0} \frac{\sin 2x}{2x^2 + x}$, 6) $\lim_{x \rightarrow \infty} \left(1 + \cos \frac{1}{x} \right)$
 7) $\lim_{x \rightarrow \infty} \frac{3x^3 + 5x^2 - 7}{10x^3 - 11x + 5}$, 8) $\lim_{y \rightarrow \infty} \frac{3y + 7}{y^2 - 2}$
 9) $\lim_{x \rightarrow \infty} \frac{x^3 - 1}{2x^2 - 7x + 5}$, 10) $\lim_{x \rightarrow -1^-} \frac{1}{x + 1}$

Sol.-

$$1) \lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} = \lim_{x \rightarrow 0} \frac{5x + 8}{3x^2 - 16} = \frac{0 + 8}{0 - 16} = -\frac{1}{2}$$

$$2) \lim_{x \rightarrow a} \frac{x^3 - a^3}{x^4 - a^4} = \lim_{x \rightarrow a} \frac{(x - a)(x^2 + ax + a^2)}{(x - a)(x + a)(x^2 + a^2)} = \frac{a^2 + a^2 + a^2}{(a + a)(a^2 + a^2)} = \frac{3}{4a}$$

$$3) \lim_{x \rightarrow 0} \frac{5 \frac{\sin 5x}{5x}}{3 \frac{\sin 3x}{3x}} = \frac{5}{3} \cdot \frac{\lim_{5x \rightarrow 0} \frac{\sin 5x}{5x}}{\lim_{3x \rightarrow 0} \frac{\sin 3x}{3x}} = \frac{5}{3}$$

$$4) \lim_{y \rightarrow 0} \frac{\tan 2y}{3y} = \frac{2}{3} \cdot \lim_{2y \rightarrow 0} \frac{\sin 2y}{2y} \cdot \lim_{y \rightarrow 0} \frac{1}{\cos 2y} = \frac{2}{3}$$

$$5) \lim_{x \rightarrow 0} \frac{\sin 2x}{2x^2 + x} = 2 \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \lim_{x \rightarrow 0} \frac{1}{2x+1} = 2$$

$$6) \lim_{x \rightarrow \infty} \left(1 + \cos \frac{1}{x} \right) = 1 + \cos 0 = 2$$

$$7) \lim_{x \rightarrow \infty} \frac{3x^3 + 5x^2 - 7}{10x^3 - 11x + 5} = \lim_{x \rightarrow \infty} \frac{3 + \frac{5}{x} - \frac{7}{x^3}}{10 - \frac{11}{x^2} + \frac{5}{x^3}} = \frac{3}{10}$$

$$8) \lim_{y \rightarrow \infty} \frac{3y+7}{y^2-2} = \lim_{y \rightarrow \infty} \frac{\frac{3}{y} + \frac{7}{y^2}}{1 - \frac{2}{y^2}} = \frac{0}{1} = 0$$

$$9) \lim_{x \rightarrow \infty} \frac{x^3 - 1}{2x^2 - 7x + 5} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{x^3}}{\frac{2}{x} - \frac{7}{x^2} + \frac{5}{x^3}} = \frac{1}{0} = \infty$$

$$10) \lim_{x \rightarrow -1^-} \frac{1}{x+1} = \frac{1}{-1+1} = -\infty$$

Ex:3

Show that (a) $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$ and (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$. Using $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

(a) Using the half-angle formula $\cos h = 1 - 2 \sin^2(h/2)$, we calculate

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} &= \lim_{h \rightarrow 0} -\frac{2 \sin^2(h/2)}{h} \\ &= -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \sin \theta \\ &= -(1)(0) = 0. \end{aligned}$$

(b)

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} &= \lim_{x \rightarrow 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\ &= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \\ &= \frac{2}{5} (1) = \frac{2}{5}\end{aligned}$$

Home Work

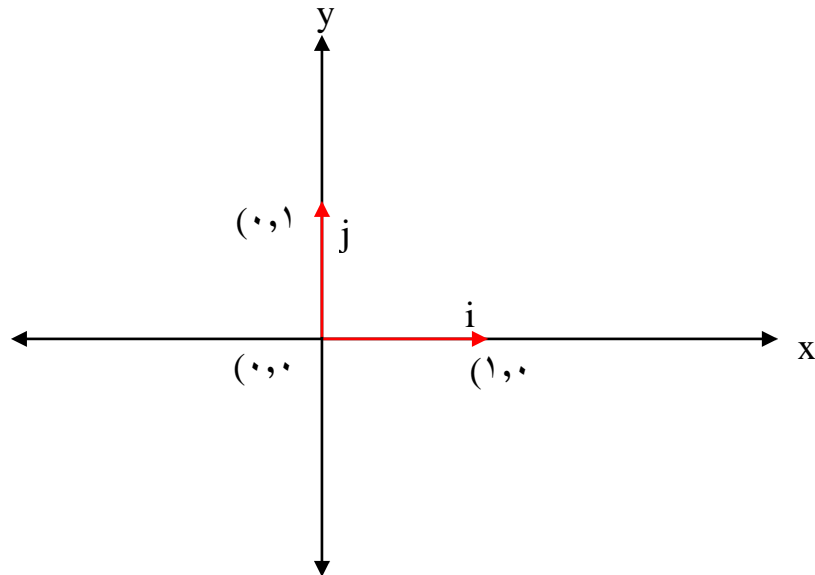
1. $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$ **Ans.(1/4)** 2. $\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2-4}}$ **Ans.(0)**
3. $\lim_{x \rightarrow 0} \frac{\sqrt{(x+1)}-1}{x}$ hint: multiply by $\frac{\sqrt{(x+1)}+1}{\sqrt{(x+1)}+1}$ **Ans. (0.5)**
4. $\lim_{x \rightarrow 0} \frac{\tan x}{x}$ **Ans.(1)**
5. $\lim_{x \rightarrow \pi/2} \frac{\cos x}{\frac{\pi}{2}-x}$ **Take $\frac{\pi}{2} - x = z$ Ans.(1)**
6. $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$ **Take $z = \frac{1}{x}$, Ans.(1)**
7. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 6x + 1} - \sqrt{x^2 + x})$ hint: multiply by $\frac{(\sqrt{x^2+6x+1}+\sqrt{x^2+x})}{(\sqrt{x^2+6x+1}+\sqrt{x^2+x})}$ **Ans.(5/2)**

Lecture 3 : Vectors**Vector components:**

Quantities can be divided into:

- 1) scalar: which is defined by magnitude only, for example mass, time...
- 2) vector: the quantity which needs magnitude and direction, for example force, velocity, acceleration....

We shall denote the vector from (x, y) to (x', y') by (i) and the vector from (x, y) to (x, y') by (j) as shown in figure below:



Then any vector in the x - y plane can be divided in the terms of i and j.

Ex. Sketch the following vectors:

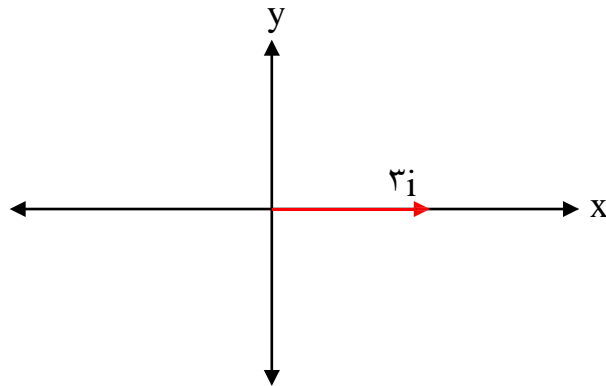
a) \hat{i}

b) $-\hat{i}$

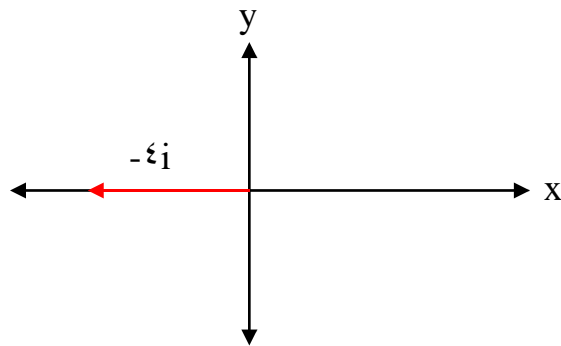
c) $-\hat{i} + \hat{j}$

Sol.:

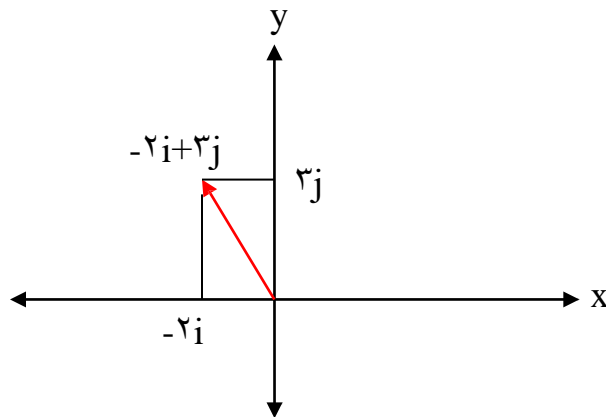
a)



b)



c)



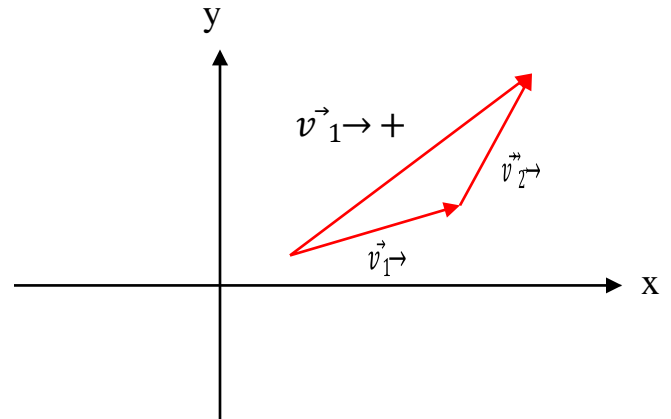
Note: when we say the vector $A \rightarrow \rightarrow B \rightarrow$ means: This vector directed from point A to point B.

Arithmetic operation on vector:

1) Addition:

Let $v_{1 \rightarrow} = a_1i + b_1j$ and $v_{2 \rightarrow} = a_2i + b_2j$

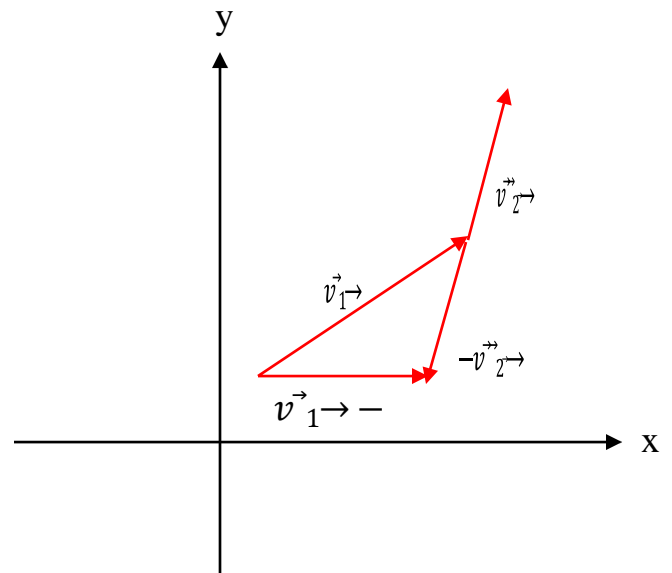
$$\begin{aligned} \text{Then } v_{1 \rightarrow} + v_{2 \rightarrow} &= a_1i + b_1j + a_2i + b_2j \\ &= (a_1 + a_2)i + (b_1 + b_2)j \end{aligned}$$



2) Subtraction:

Let $v_{1 \rightarrow} = a_1i + b_1j$ and $v_{2 \rightarrow} = a_2i + b_2j$

$$\begin{aligned} \text{Then } v_{1 \rightarrow} - v_{2 \rightarrow} &= (a_1i + b_1j) - (a_2i + b_2j) \\ &= (a_1 - a_2)i + (b_1 - b_2)j \end{aligned}$$



3) Multiplication of vector by scalar:

Let $v = ai + bj$ and c is scalar.

$$\text{then } cv = c(ai + bj) = cai + cbj$$

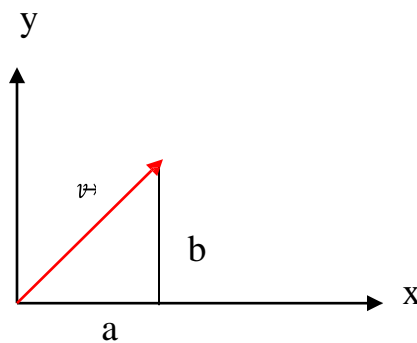
Definitions:**1) Length of vector:**

The length of vector $\vec{v} = ai + bj$ is usually denoted by $|\vec{v}|$ which may be read as "The magnitude of \vec{v} ".

$|\vec{v}|$ is found using Pythagoras theorem and as follows:

$$\vec{v} = ai + bj$$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$

**2) Zero vector:**

Any vector of length zero is called zero vector $\vec{0}$,

$$ai + bj = 0$$

$$ai + bj = 0i + 0j \text{ if and only if } a = b = 0.$$

3) Unit vector:

It is part from any vector. This part has length equal to unity and it is used to describe the direction of the vector.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} \quad \text{where } \vec{u} \text{ is unit vector of } \vec{v}.$$

Ex. Find the unit vector of $\vec{v} = ai + bj$ and prove that unit vector depends on the angle between the vector and the x - axis?

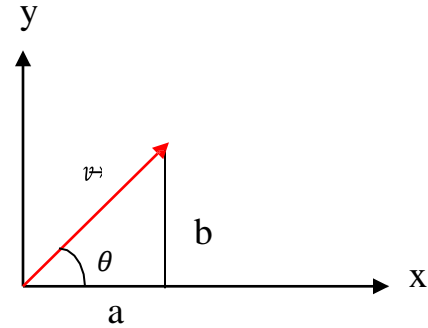
Sol.:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{u} = \frac{ai + bj}{\sqrt{a^2 + b^2}}$$

$$\vec{u} = \frac{a}{\sqrt{a^2 + b^2}}i + \frac{b}{\sqrt{a^2 + b^2}}j$$

$$\vec{u} = \cos\theta i + \sin\theta j$$



Ex. Find the unit vector of $\vec{A} = 3i + 4j$.

Sol.:

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|}$$

$$|\vec{A}| = \sqrt{3^2 + 4^2} = 5$$

$$\vec{u} = \frac{3i + 4j}{5} = 0.6i + 0.8j$$

since

$$\vec{u} = \cos\theta i + \sin\theta j$$

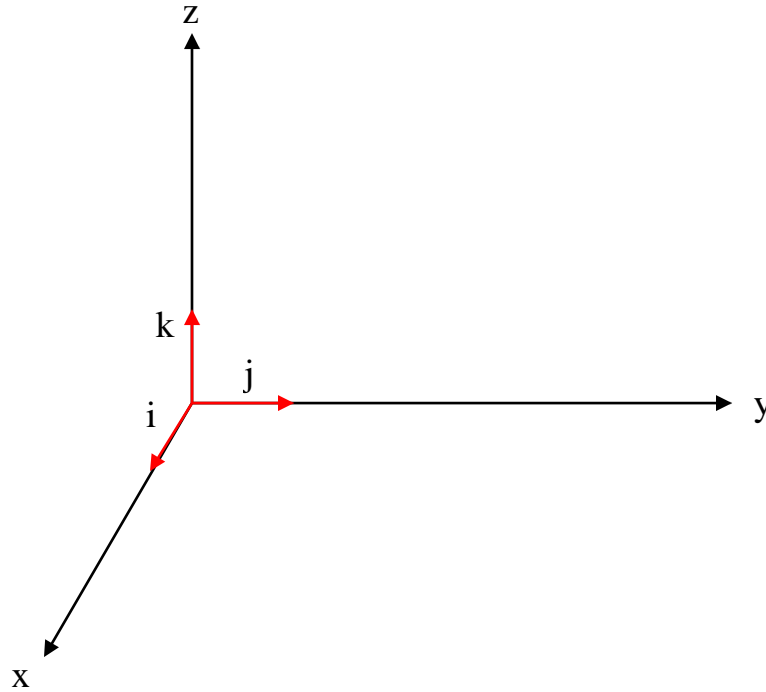
$$\cos\theta = 0.6$$

$$\therefore \theta = 53^\circ$$

Vector in Space:

Now we shall consider the vector in three dimensional space as follows:

- (i) as a vector pointing from (x, y, z) to (x', y', z')
- (j) as a vector pointing from (x, y, z) to (x, y', z)
- (k) as a vector pointing from (x, y, z) to (x, y, z')



Any vector $A \rightarrow$ for example may be represented as:

$$A \rightarrow = ai + bj + ck$$

and

$$|A \rightarrow| = \sqrt{a^2 + b^2 + c^2}$$

Ex.: Find a unit vector in the direction of vector from $P_1(1, 1, 1)$ and $P_2(3, 2, 1)$.

Sol.:

$$\vec{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 1)\mathbf{j} + (1 - 1)\mathbf{k}$$

$$\vec{P_1P_2} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$|\vec{P_1P_2}| = \sqrt{4 + 1 + 1} = \sqrt{6} = 2.45$$

$$\hat{u} = \frac{\vec{P_1P_2}}{|\vec{P_1P_2}|} = \frac{2\mathbf{i} + \mathbf{j} - \mathbf{k}}{2.45} = \frac{2}{2.45}\mathbf{i} + \frac{1}{2.45}\mathbf{j} - \frac{1}{2.45}\mathbf{k}$$

Scalar product (dot product):

The scalar or dot product of two vectors \vec{A} and \vec{B} , denoted by $\vec{A} \cdot \vec{B}$ (read \vec{A} dot \vec{B}), is defined as the product of the magnitudes of \vec{A} and \vec{B} and the cosine of the angle θ between them.

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos\theta, \quad 0 \leq \theta \leq \pi$$

The following laws are valid:

$$1. \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

$$2. \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

$$3. \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

The dot product can be used to find:

1) The angle between two vectors.

2) The projection of vector \vec{B} on $\vec{A} = |\vec{B}| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|}$

3) The projection of \vec{A} on $\vec{B} = |\vec{A}| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$

Ex. Find the angle between $\vec{A} = i - 2j - 2k$ and $\vec{B} = 6i + 3j + 2k$ also find the projection of \vec{A} on \vec{B} and \vec{B} on \vec{A} .

Sol.:

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos\theta$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$\vec{A} \cdot \vec{B} = 1 \cdot 6 + (-2) \cdot 3 + (-2) \cdot 2 = -4$$

$$|\vec{A}| = \sqrt{1 + 4 + 4} = 3$$

$$|\vec{B}| = \sqrt{36 + 9 + 4} = 7$$

$$\cos\theta = \frac{-4}{3 \cdot 7} = \frac{-4}{21}$$

$$\theta = 101^\circ$$

The projection of $A \rightarrow$ on $B \rightarrow = |A \rightarrow| \cos \theta = \frac{A \rightarrow \cdot B \rightarrow}{|B|} = \frac{-4}{\sqrt{5}}$

2) The projection of vector $B \rightarrow$ on $A \rightarrow = |B \rightarrow| \cos \theta = \frac{A \rightarrow \cdot B \rightarrow}{|A|} = \frac{-4}{\sqrt{5}}$

Orthogonal vectors:

The two vectors $A \rightarrow$ and $B \rightarrow$ are orthogonal if and only if:

$$A \rightarrow \cdot B \rightarrow = 0$$

Lec : 3 Derivatives of trigonometric functions

If u is a function x , the chain rule version of this differentiation rule is

$$\begin{aligned}
 1. \quad \frac{d}{dx} (\sin u) &= \cos u \cdot \frac{du}{dx} \\
 2. \quad \frac{d}{dx} (\cos u) &= -\sin u \cdot \frac{du}{dx} \\
 3. \quad \frac{d}{dx} (\tan u) &= \sec^2 u \cdot \frac{du}{dx} \\
 4. \quad \frac{d}{dx} (\cot u) &= -\csc^2 u \cdot \frac{du}{dx} \\
 5. \quad \frac{d}{dx} (\sec u) &= \sec u \tan u \cdot \frac{du}{dx} \\
 6. \quad \frac{d}{dx} (\csc u) &= -\csc u \cot u \cdot \frac{du}{dx}
 \end{aligned}$$

Example 1: Find derivatives of the functions

$$1. \quad y = \sin^2 x \quad \Leftrightarrow \quad y = (\sin x)^2 \quad \Leftrightarrow \quad \frac{dy}{dx} = 2 \sin x \cos x = \sin 2x$$

$$2. \quad y = \cos(x^2) \quad \Leftrightarrow \quad \frac{dy}{dx} = -2x \sin(x^2)$$

$$3. \quad y = \tan \sqrt{x} \quad \Leftrightarrow \quad \frac{dy}{dx} = \sec^2 \sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{\sec^2 \sqrt{x}}{2\sqrt{x}}$$

$$4. \quad y = x^2 \sec 3x \quad \Leftrightarrow \quad \frac{dy}{dx} = 3x^2 \sec 3x \tan 3x + 2x \sec 3x$$

$$= (x \sec 3x)(2 + 3x \tan 3x)$$

$$\begin{aligned}
 5. \quad y = \sqrt{\sin 2x} \quad \Leftrightarrow \quad y = (\sin 2x)^{1/2} \quad \Leftrightarrow \quad \frac{dy}{dx} &= \frac{1}{2} (\sin 2x)^{-1/2} \times \cos 2x \times 2 \\
 &= \frac{\cos 2x}{\sqrt{\sin 2x}}
 \end{aligned}$$

Example 2: If $y = \tan 2t$ and $x = \sec 2t$ show that $\frac{dy}{dx} = \csc 2t$

$$\begin{aligned}\frac{dy}{dt} &= 2 \sec^2 2t \quad , \quad \frac{dx}{dt} = 2 \sec 2t \tan 2t \\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = 2 \sec^2 2t \times \frac{1}{2 \sec 2t \tan 2t} = \frac{\sec 2t}{\tan 2t} \\ &= \frac{1}{\frac{\cos 2t}{\sin 2t}} = \frac{1}{\frac{\cos 2t}{\sin 2t}} = \csc 2t\end{aligned}$$

Example 3: If $y = \theta - \cos \theta$ and $x = \theta + \cos \theta$; ($0 \leq \theta \leq \frac{\pi}{2}$) show that $\frac{dy}{dx}$

$$= (\sec \theta + \tan \theta)^2$$

$$\frac{dy}{d\theta} = 1 + \sin \theta \quad \text{and} \quad \frac{dx}{d\theta} = 1 - \sin \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{1 + \sin \theta}{1 - \sin \theta}$$

$$\frac{dy}{dx} = \frac{1 + \sin \theta}{1 - \sin \theta} \times \frac{1 + \sin \theta}{1 + \sin \theta} = \frac{1 + 2 \sin \theta + \sin^2 \theta}{1 - \sin^2 \theta} = \frac{1 + 2 \sin \theta + \sin^2 \theta}{\cos^2 \theta}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 \theta} + \frac{2 \sin \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \sec^2 \theta + 2 \sec \theta \tan \theta + \tan^2 \theta = (\sec \theta + \tan \theta)^2$$

Inverse trigonometric functions

The inverse trigonometric functions are defined to be the inverses of particular parts of the trigonometric functions; parts that do have inverses. The inverse sine function, denoted by $\sin^{-1} x$ (some books use the notation $\arcsin(x)$), is defined to be the inverse of the restricted sine function. A similar idea holds for all the other inverse trigonometric functions. It is important here to note that in this case the “ -1 ” is not an exponent and so,

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

In inverse trigonometric functions the “ -1 ” looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trigonometric function. It is a notation that we use in this case to denote inverse trigonometric functions. If we had really wanted exponentiation to denote 1 over sine, we would use the following:

$$(\sin x)^{-1} = \frac{1}{\sin x}$$

Derivatives of inverse trigonometric functions

Let u be a function x , the derivatives of inverse trigonometric functions are:

$$1. \frac{d}{dx} (\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$2. \frac{d}{dx} (\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$3. \frac{d}{dx} (\tan^{-1} u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$4. \frac{d}{dx} (\cot^{-1} u) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$$

$$5. \frac{d}{dx} (\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$6. \frac{d}{dx} (\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Example 4: Find the derivative for

$$1. y = \sin^{-1} 2x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - (2x)^2}} \times 2 = \frac{2}{\sqrt{1 - 4x^2}}$$

$$2. y = 3x \cos^{-1} 3x - \sqrt{1 - 9x^2}$$

$$\begin{aligned} \frac{dy}{dx} &= 3x \times \frac{-1}{\sqrt{1 - (3x)^2}} \times 3 + 3 \cos^{-1} 3x - \frac{-18x}{2\sqrt{1 - 9x^2}} \\ &= \frac{-9x}{\sqrt{1 - 9x^2}} + 3 \cos^{-1} 3x + \frac{9x}{\sqrt{1 - 9x^2}} = 3 \cos^{-1} 3x \end{aligned}$$

$$3. y = 2\sqrt{x} \tan^{-1} \sqrt{x}$$

$$\frac{dy}{dx} = 2\sqrt{x} \times \frac{1}{1 + (\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} + \tan^{-1} \sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{1}{1 + x} + \frac{\tan^{-1} \sqrt{x}}{\sqrt{x}}$$

Home Work :

Find derivative in each of the following problems(1 - 4)

1. $y = \sec^2 2x$

2. $y = x^2 \sin x + 2x \cos x - 2 \sin x$

3. $y = \sqrt{x^2 - 1} - \sec^{-1} x$

4. $y = 2x \cos^{-1} \sqrt{x} + \sin^{-1} \sqrt{x} - 2\sqrt{x - x^2}$

5. If $y = 1 - \sin \theta$ and $x = \theta - \sin \theta$ find $\frac{dy}{dx}$

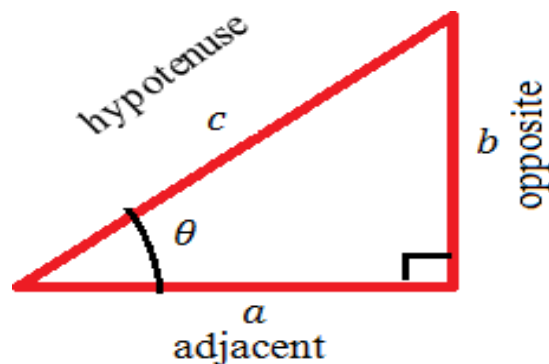
6. If $y = \sec^{-1} t$ and $x = \sqrt{t^2 - 1}$ find $\frac{dy}{dx}$

Lecturer 3 : Trigonometric functions and some applications

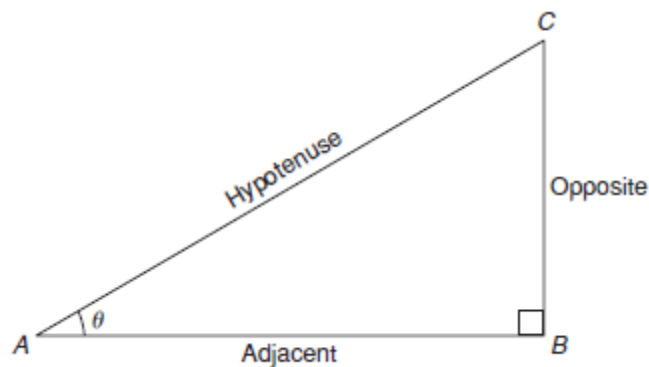
Trigonometric functions

There are six basic trigonometric functions used in Trigonometry. These functions are trigonometric ratios. The six basic trigonometric functions are sine function, cosine function, secant function, co-secant function, tangent function, and co-tangent function. The trigonometric functions and identities are the ratio of sides of a right-angled triangle. The sides of a right triangle are the perpendicular side, hypotenuse, and base, which are used to calculate the sine, cosine, tangent, secant, cosecant, and cotangent values using trigonometric formulas.

A right triangle is a triangle with a right angle (90°)



For every angle θ in the triangle, there is the side of the triangle adjacent to it, the side opposite of it and the hypotenuse such that $a^2 + b^2 = c^2$.



$$\text{sine } \theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

'Sine' is abbreviated to 'sin', thus $\sin \theta = \frac{BC}{AC}$

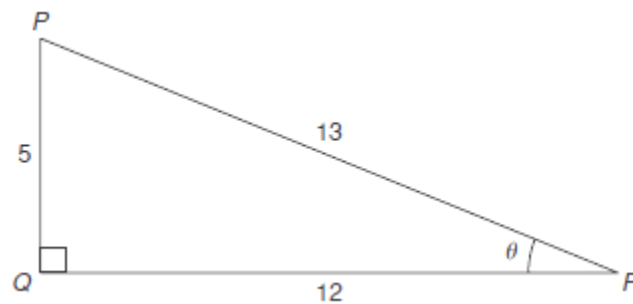
Also, $\text{cosine } \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$

'Cosine' is abbreviated to 'cos', thus $\cos \theta = \frac{AB}{AC}$

Finally, $\text{tangent } \theta = \frac{\text{opposite side}}{\text{adjacent side}}$

'Tangent' is abbreviated to 'tan', thus $\tan \theta = \frac{BC}{AB}$

EX 1 : In triangle PQR determine $\sin \theta$, $\cos \theta$ and $\tan \theta$



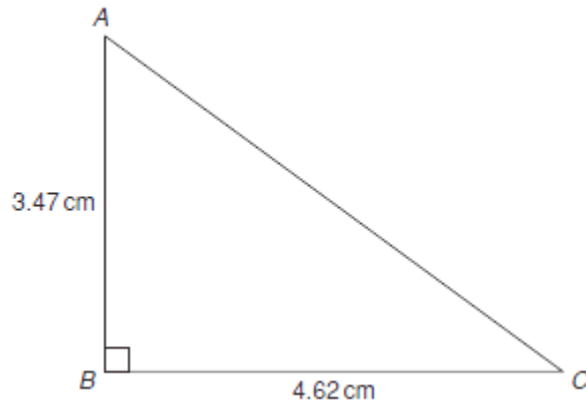
Sol:

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PQ}{PR} = \frac{5}{13} = 0.3846$$

$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{QR}{PR} = \frac{12}{13} = 0.9231$$

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PQ}{QR} = \frac{5}{12} = 0.4167$$

EX : 2 / In triangle ABC of Figure , determine length AC , $\sin C$, $\cos C$, $\tan C$, $\sin A$, $\cos A$ and $\tan A$



By Pythagoras, $AC^2 = AB^2 + BC^2$

i.e. $AC^2 = 3.47^2 + 4.62^2$

from which $AC = \sqrt{3.47^2 + 4.62^2} = 5.778\text{cm}$

$$\sin C = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{AB}{AC} = \frac{3.47}{5.778} = 0.6006$$

$$\cos C = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{4.62}{5.778} = 0.7996$$

$$\tan C = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{AB}{BC} = \frac{3.47}{4.62} = 0.7511$$

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{4.62}{5.778} = 0.7996$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{AB}{AC} = \frac{3.47}{5.778} = 0.6006$$

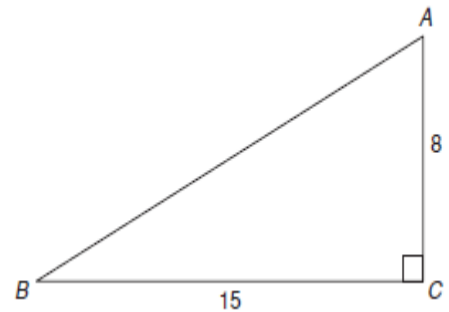
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{BC}{AB} = \frac{4.62}{3.47} = 1.3314$$

EX : 3 / If $\tan B = \frac{8}{15}$, determine the value of $\sin B$, $\cos B$, $\sin A$ and $\tan A$

SOL:

A right-angled triangle ABC is shown in figure If

$\tan B = \frac{8}{15}$, then $AC = 8$ and $BC = 15$.



By Pythagoras, $AB^2 = AC^2 + BC^2$

i.e. $AB^2 = 8^2 + 15^2$

from which $AB = \sqrt{8^2 + 15^2} = 17$

$$\sin B = \frac{AC}{AB} = \frac{8}{17} \text{ or } 0.4706$$

$$\cos B = \frac{BC}{AB} = \frac{15}{17} \text{ or } 0.8824$$

$$\sin A = \frac{BC}{AB} = \frac{15}{17} \text{ or } 0.8824$$

$$\tan A = \frac{BC}{AC} = \frac{15}{8} \text{ or } 1.8750$$

the trigonometric functions are defined as follows:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hyp}}{\text{adj}} = \frac{c}{a}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hyp}}{\text{opp}} = \frac{c}{b}$$

Trigonometric functions of negative angles

$$\sin(-\theta) = -\sin \theta \quad , \quad \cos(-\theta) = \cos \theta \quad \text{and} \quad \tan(-\theta) = -\tan \theta$$

Some useful relationships among trigonometric functions

1. $\sin^2x + \cos^2x = 1$, $\sec^2x - \tan^2x = 1$, $\csc^2x - \cot^2x = 1$

2. $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2x - \sin^2x = 1 - 2\sin^2x = 2\cos^2x - 1$

3. $\sin^2x = \frac{1 - \cos 2x}{2}$, $\cos^2x = \frac{1 + \cos 2x}{2}$

Lecture 5: Partial Derivatives

1. Introduction

In many real-life applications, functions depend on more than one variable such as $f(x, y)$ or $f(x, y, z)$. To understand how the function changes when only one variable changes while all others remain constant, we use partial derivatives.

2. Definition

If $f(x, y)$ is a function of two variables, then the partial derivative with respect to x is:

$$\frac{\partial f}{\partial x}$$

Meaning: differentiate with respect to x while treating y as a constant.

Similarly,

$$\frac{\partial f}{\partial y}$$

is the derivative with respect to y .

Example 1:

$$f(x, y) = x^2 y + 5y$$

$$\frac{\partial f}{\partial x} = 2xy$$

$$\frac{\partial f}{\partial y} = x^2 + 5$$

Example 2:

$$f(x, y) = \sin(xy)$$

$$\frac{\partial f}{\partial x} = \cos(xy) * y$$

$$\frac{\partial f}{\partial y} = \cos(xy) * x$$

Example 3:

$$f(x, y, z) = x^2 + yz + e^{xz}$$

$$f_x = 2x + z e^{xz}$$

$$f_y = z$$

$$f_z = y + x e^{xz}$$

Ex.: Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of the following functions:

a. $z = x^2 + 3xy + y - 1$

b. $z = \ln(x^2 - y)$

c. $z = x \cos(y) + y e^x$

d. $z = y \sin(xy)$

Sol.:

a. To find $\frac{\partial z}{\partial x}$ treat y as a constant and differentiate with respect to x. We

have $z = x^2 + 3xy + y - 1$ so:

$$\frac{\partial z}{\partial x} = 2x + 3y$$

Similarly

$$\frac{\partial z}{\partial y} = 3x + 1$$

$$\mathbf{b.} \quad z = \ln(x^2 - y)$$

$$\frac{\partial z}{\partial x} = \frac{1}{(x^2 - y)} 2x = \frac{2x}{(x^2 - y)}, \quad \frac{\partial z}{\partial y} = \frac{-1}{(x^2 - y)}$$

$$\mathbf{c.} \quad z = x \cos(y) + ye^x$$

$$\frac{\partial z}{\partial x} = \cos(y) + ye^x, \quad \frac{\partial z}{\partial y} = -x \sin(y) + e^x$$

$$\mathbf{d.} \quad z = y \sin(xy)$$

$$\frac{\partial z}{\partial x} = y \cos(xy) y = y^2 \cos(xy)$$

$$\frac{\partial z}{\partial y} = x y \cos(xy) + \sin(xy)$$

2- Derivatives of the Natural Logarithm Function (ln)

1. Derivative of ln(x) in One Variable

$$d/dx \ln(x) = \frac{1}{x}, \quad x > 0$$

- Example:

$$\frac{d}{dx} \ln(3x^2 + 1) = (1 / (3x^2 + 1)) * 6x = \frac{6x}{(3x^2 + 1)}$$

3- Partial Derivative of a Multivariable ln Function

If we have $f(x, y) = \ln(xy + y^2)$, the partial derivatives are:

$$1. \quad \partial f / \partial x$$

$$f = \ln(u), \quad u = xy + y^2$$

$$\partial f / \partial x = 1/u * \partial u / \partial x = 1/(xy + y^2) * y = y / (xy + y^2)$$

4- More Complex In Function

Example: $f(x, y, z) = \ln(xy + z^2)$

Partial derivative w.r.t x: $\partial f / \partial x = 1/(xy + z^2) * y = y / (xy + z^2)$

Partial derivative w.r.t y: $\partial f / \partial y = 1/(xy + z^2) * x = x / (xy + z^2)$

Partial derivative w.r.t z: $\partial f / \partial z = 1/(xy + z^2) * 2z = 2z / (xy + z^2)$

5- Derivative of e^x in One Variable

$$d/dx e^x = e^x$$

- Basic rule: the derivative of e^x is itself.

- Example:

$$d/dx e^{3x} = 3 e^{3x}$$

Partial Derivative of a Multivariable Exponential Function

If we have $f(x, y) = e^{xy + y^2}$, the partial derivatives are:

1. $\partial f / \partial x$

$$f = e^u, u = xy + y^2$$

$$\partial f / \partial x = e^u * \partial u / \partial x = e^{xy + y^2} * y = y e^{xy + y^2}$$

 $\partial f / \partial y$

$$\partial u / \partial y = x + 2y$$

$$\partial f / \partial y = e^u * (x + 2y) = (x + 2y) e^{xy + y^2}$$

More Complex Exponential Functions

Example: $f(x, y, z) = e^{xy + z^2}$

Partial derivative w.r.t x: $\partial f / \partial x = e^{(xy + z^2)} * y = y e^{(xy + z^2)}$

Partial derivative w.r.t y: $\partial f / \partial y = e^{(xy + z^2)} * x = x e^{(xy + z^2)}$

Partial derivative w.r.t z: $\partial f / \partial z = e^{(xy + z^2)} * 2z = 2z e^{(xy + z^2)}$