lec. 1 / introduction to integration

Basic rules of integration

① $\int a \cdot dx = ax + c$

Ex| $\int 3 \cdot dx = 3x + c$

② $\int ax \cdot dx = a \int x \cdot dx = a \frac{x^2}{2} + c$

Ex| $\int 5x^4 \cdot dx = 5 \int x^4 \cdot dx = 5 \frac{x^5}{5} + c$

$= x^5 + c$

③ $\int [f(x) + g(x)] \cdot dx$

$= \int f(x) \cdot dx + \int g(x) \cdot dx$

Ex| $\int (4x^2 + 5x^6) \cdot dx$

$= \int 4x^2 \cdot dx + \int 5x^6 \cdot dx$

$= \frac{4}{3}x^3 + \frac{5}{7}x^7 + c$

④ $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$

Ex| $\int x^3 dx = \frac{x^4}{4} + c$

⑤ $\int \sin x \cdot dx = -\cos x + c$

⑥ $\int \cos x \cdot dx = \sin x + c$

Ex1/ $\int \sin x \cos x \cdot dx$ — مشتقة الزاوية — وهو تكون مشتقة الثاني

$$= \frac{\sin^2 x}{2} + c$$

Ex/ $\int \sin 3x \cdot dx$ ← لازم نوفر مشتقة الزاوية

$$= \frac{1}{3} \int \sin 3x \cdot 3 \cdot dx$$

$$= \frac{1}{3} * -\cos 3x + c$$

$$= -\frac{1}{3} \cos 3x + c$$

⑦ $\int e^{3x} \cdot dx$

$$= \frac{1}{3} e^{3x} + c$$

مشتقة الاس

⑦ $\int \frac{1}{x} \cdot dx$

$$= \ln |x| + c$$

Integration by Parts.

هذه الطريقة عندما يكون لدينا تكامل دالتين ولكن لا يمكن ان تكون احداهما مشتقة الأخرى

Ex| $\int x \cos x . dx$

نستخدم طريقة $(u.dv)$

① نفرض u و dv

ملاحظة/ u دائماً هي الدالة التي يكون ناتج اشتقاقها النهائي 0

مشتقة | تكامل

$u = x \qquad dv = \cos x$

$du = 1\, dx \qquad v = \sin x$

$$\int u . dv = u . v - \int v . du$$

$= x \sin x - \int \sin x . dx$

$= x \sin x + \cos x + c$

Ex| $\int x . e^x dx$

let $u = x \qquad dv = e^x$

$du = 1 \qquad v = e^x$

* اخذنا x هي u لانها هي دالة متعددة ناتج اشتقاقها هو (0)

* تكامل الـ e هي الدالة نفسها مع توفير مشتقة اسها

$\int x . e^x = x e^x - \int e^x . dx$

$\int x . e^x = x e^x - e^x + c$

طلبا منك تحلها بطريقة
Par

Ex/ $\int \ln x \cdot dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} \cdot dx \qquad v = x$

$= x \ln x - \int x \cdot \frac{1}{x} \cdot dx$

$= x \ln x - x + c$

Ex/ $\int x^2 e^{-x}$

هنا انتبه انا x^2 يعني راح اشغل الشغل مرتين

$u = x^2 \qquad dv = e^{-x}$

$du = 2x \cdot dx \qquad v = -e^{-x}$

$= -x^2 e^{-x} - \int 2x \cdot (-e^{-x}) \cdot dx$

$= -x^2 e^{-x} + 2 \int x e^{-x} \cdot dx$

$u = x \qquad dv = e^{-x}$

$du = 1 \qquad v = -e^{-x}$

$= -x^2 e^{-x} + 2 [-x e^{-x} - \int -e^{-x} \cdot dx$

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c$

الحل بطريقة آخرى (تابلور
هذه الطريقة فقط
اذا كان الاس اكبر من (1)
(2) كما فوق

$\int x^2 \cdot e^{-x}$

$u = x^2 \quad + \quad dv = e^{-x}$

x^2	$+$	e^{-x}
$2x$	$-$	$-e^{-x}$
2	$+$	e^{-x}
0		$-e^{-x}$

$$= -x^2 e^{-x} - 2x e^{-x} - 2 e^{x}$$

Ex1 $\int x^4 e^{3x}\, dx$

بما ان الاس اكبر من 2 نستخدم
طريقة ايا بلوز)

$u = x^4 \quad + \quad dv = e^{3x}$

x^4	$+$	e^{3x}
$4x^3$	$-$	$\frac{1}{3}e^{3x}$
$12x^2$	$+$	$\frac{1}{9}e^{3x}$
$24x$	$-$	$\frac{1}{27}e^{3x}$
24	$+$	$\frac{1}{81}e^{3x}$
0		$\frac{1}{243}e^{3x}$

$$= \frac{x^4 e^{3x}}{3} - \frac{4x^3 e^{3x}}{9} + \frac{4x^2 e^{3x}}{9} - \frac{8x e^{3x}}{27}$$

$$+ \frac{8 e^{3x}}{81} + C$$

$\frac{12}{27} = \frac{4}{9}$

$\frac{24}{81} = \frac{8}{27}$

$\frac{24}{243}$

lec.2/ Integration of Rational Functions by partial fraction

تكامل الدوال النسبية بواسطة تجزئة الكسور

شروط استخدام هذه الطريقة :

(١) ان تكون الدالة كسرية

(٢) كل من البسط والمقام متعددة الحدود

(٣) ان تكون درجة البسط اقل من المقام

(٤) ان يكون المقام قابل للتحليل الى عوامل على هيئة $(ax^2+bx+c)^m$ و $(ex-d)^k$

Ex.1

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$$

الحل

(١) هنا درجة البسط اقل من درجة المقام
لان اذا ضربنا المقام راح نحصل x^3
اي اعلى من البسط الذي هو x^2

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}$$

نكون فرضية

نعوض حتى نحصل قيم A و B و C
لازم نعوض بالمقام بحيث يكون المقام
صفر يعني قيمة A نعوض فيه $x=1$

$x=1 \Rightarrow A =$

$$\frac{x^2+4x+1}{\boxed{(x-1)}(x+1)(x+3)}$$

نغطي الجزء

ونعوض بالباقي $x=1$

$$\frac{(1)^2+4\times1+1}{(1+1)(1+3)} = \frac{6}{8} = \frac{3}{4}$$

قيمة $B/\quad x=-1$ ونعوض

$$\frac{x^2+4x+1}{(x-1)\boxed{(x+1)}(x+3)}$$

cover نغطي

ونعوض بالباقي بمكان $x=-1$

$$\frac{(-1)^2+4\times(-1)+1}{(-1-1)(-1+3)} = \frac{-2}{-4} = \frac{1}{2}$$

③ $x = -3$ وشوش

$$\frac{x^2+4x+1}{(x-1)(x+1)\boxed{(x+3)}}$$

نغطي ونعوض

$$\frac{(-3)^2+4*(-3)+1}{(-3-1)(-3+1)} = \frac{-2}{8} = -\frac{1}{4}$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx = \int \frac{\frac{3}{4}}{(x-1)}dx + \int \frac{1/2}{(x+1)}dx + \int \frac{1/4}{(x+3)}.dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + C$$

مشتقه المقام موجوده اذا التكامل هو ln

EX] $\int \frac{6x+7}{(x+2)^2}.dx$

ملاحظه/ لما ان عندي المقام اسه تربيع سوف نقوم نفتح المقام الى جزئين $(x+2)$ و $(x+2)^2$

واذا كان الاس اكبر سوف نقوم نفتحه لعدد الاكبر مثلا $(x+2)^3$ سوف يصبح $(x+2)\ (x+2)^2\ (x+2)^3$ وهكذا

$$\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} \quad \text{--①}$$

نستخرج قيمه B بطريقه cover نعوض بمكان كل $x=-2$

$$\frac{6x+7}{\boxed{(x+2)^2}} = 6*(-2)+7 = \boxed{-5}$$

نغطي المقام والجوهر فقط بالبسط

ملاحظه/ لما ان طريقه ال cover تحتاج تغطي واحد من الاجزاء وليس بالمعادله الاصليه الاصعام B نستطيع استخراج قيمه B فقط بطريقه التغطيه اما قيمه A يتم استخراجها بطريقه التعويض

لاستخراج قيمه A سوف افرض اي قيمه لا X ولذلك $x=0$ ولنعوض بطرفين المعادله (1) مع تعويض قيمه B

$$\frac{6*0+7}{(0+2)^2} = \frac{A}{(0+2)} + \frac{-5}{(0+2)^2}$$

نوحد مقامات نضرب * 4

$$\frac{7}{4} = \frac{A}{2} - \frac{5}{4}$$

$$7 = 2A - 5 \Rightarrow 2A = 12 \Rightarrow A = 6$$

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \frac{6}{(x+2)}\,dx + \int \frac{-5}{(x+2)^2}\,dx$$

$$\int (x+2)^{-2} = -5\,(x+2)^{-1} +$$

$$= 6\ln|x+2| - 5\cdot(-(x+2)^{-1}) + c$$

$$= 6\ln|x+2| + \frac{5}{(x+2)} + c$$

* * *

EX1

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

لما انا عندي المقام الاول بيه x^2
لازم قيمه A نعوض بطريقه $Ax+B$
وبس قيمه A فقط اما قيمه C
و D نكونا فقط وحده لان الاس X هوا 1

$$\frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$

لان الاس $(x-1)^2$

نخرج قيمه D فقط لانها الوحيد الموجود بمقامها وحد
طريقه القطعه هنا سوف نستخرج قيمه
نعوض قيمه x التي تجعل المقام 0 وهي $x=1$

$$x=1 \;/\; D = \frac{-2*(1)+4}{((1)^2+1)} = \frac{2}{2} = \boxed{1 = D}$$

هنا نخرج قيمه نعم A و B و C سوف نعوض قيم عشوائيه ل x ونخرج منها معادلات
ونجمعها ونخرج قيمه الباقي

① let $x=0$

$$\frac{-2*(0)+4}{(0+2)(0-1)^2} = \frac{A*0+B}{(0+1)} + \frac{C}{(0-1)} + \frac{1}{(0-1)^2}$$

مقامها -1

$$4 = B - C + 1 \Rightarrow \boxed{B - C = 3} \quad ---①$$

let $x = -1$

$$\frac{-2 * -1 + 4}{((-1)^2+1)(-1-1)^2} = \frac{A * -1 + B}{(-1)^2 + 1} + \frac{C}{(-1-1)} + \frac{1}{(-1-1)^2}$$

$$\frac{6}{8} = \frac{-A + B}{2} + \frac{C}{-2} + \frac{1}{4}$$

$$3 = -2A + 2B - 2C + 1$$

$$2 = -2A + 2B - 2C$$

$$\boxed{1 = -A + B - C} \quad \text{--- ②}$$

let/ $x = 2$

$$\frac{-2 * 2 + 4}{(2^2+1)(2-1)^2} = \frac{A * 2 + B}{2^2 + 1} + \frac{C}{(2-1)} + \frac{1}{(2-1)^2}$$

$$0 = \frac{2A + B}{5} + \frac{C}{1} + \frac{1}{1}$$

$$0 = 2A + B + 5C + 5 \Rightarrow \boxed{2A + B + 5C = -5} \quad \text{--- ③}$$

نتخلص من A

$$-A + B - C = 1 \quad * 2$$
$$2A + B + 5C = -5$$

$$-2A + 2B - 2C = 2$$
$$2A + B + 5C = -5$$

بالجمع

$$3B + 3C = -3 \quad \text{--- ④}$$

نقسم على ٣

$$B + C = -1$$

معادلة ١

$$B - C = 3$$

بالجمع

$$2B = 2 \Rightarrow B = 1$$

$C = -2$

$A =$

$-A + B - C = 1$ sub C and B

$-A + 1 + 2 = 1$

$-A + 3 = 1 \Rightarrow -A = -2 \Rightarrow A = 1$

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2} dx = \int \frac{x+1}{(x^2+1)} dx + \int \frac{-2}{(x-1)} dx + \int \frac{1}{(x-2)^2} dx$$

→ H.w

=

H.w 2 / $\int \frac{dx}{x(x^2+1)^2}$

lec 3 / Integration by compleate the sequare ①

طريقة التكامل بإكمال المربع.

ملاحظة / كيف نكمل المربع

الشروط اكمال المربع

① ان يكون معامل $x^2 = 1$

② نستخرج معامل $x = 2$

③ نصف معامل $x = 1$

④ نضيف (نصف معامل x)² للحد ∓

$(1)^2 = 1$

EX1 $x^2 + 2x + \cdots$

هذا المربع الكامل

$x^2 + 2x + 1 - 1$

$(x + 1)^2 - 1$

جذر الاول لفس الاشارة وجذر الحد الاخير

الشرط الاول محقق

① معامل $x^2 = 1$

② معامل $x = 1$

③ نصف معامل $x = \frac{1}{2}$

④ تربيع معامل $x = (\frac{1}{2})^2 = \frac{1}{4}$

EX1 $x^2 + x + \cdots$

$x^2 + x + \frac{1}{4} - \frac{1}{4}$

$(x + \frac{1}{2})^2 - \frac{1}{4}$

جذر الاول لفس الاشارة جذر الاخير

$(x + \frac{1}{2})^2 - \frac{1}{4}$

---

① لازم معامل $x^2 = 1$

② معامل $x = 3$

نصف معامل $x = \frac{3}{2}$

تربيع $(\frac{3}{2})^2 = \frac{9}{4}$

نضيف للحد

EX1 $4x^2 + 12x + \cdots$

سحب 4 عامل مشترك

$4(x^2 + 3x + \cdots)$

$4(x^2 + 3x + \frac{9}{4} - \frac{9}{4})$

$4[(x + \frac{3}{2})^2 - \frac{9}{4}]$

Ex1 Evaluate the integration.

هنا لايوجد مشتقه المقام
ولايمكن حله بالطرق الاعتياديه
اذا نستخدم طريقه اكمال المربع

$$\int \frac{dx}{\sqrt{x^2 - 2x + 17}}$$

رقم ١٧
لايكمل المربع

$$= \int \frac{dx}{\sqrt{x^2 - 2x + 1 - 1 + 17}}$$

مربع كامل

ننزل

① معامل $x^2 = 1$
② معامل $x = 2$
③ نصف معامل $x = 1$
تربيع ، $(1)^2 = 1$
نضيف $+1 -1$

$$= \int \frac{dx}{\sqrt{(x-1)^2 + 16}}$$

هذا يمثل تكامل الدال العكسي

$$\frac{du}{\sqrt{u^2 + a^2}} = \sinh^{-1} \frac{u}{a} + c$$

let $u^2 = (x-1)^2 \Rightarrow u = (x-1)$

$a^2 = 16 \Rightarrow a = 4$

$dx = du$

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \sinh^{-1} \frac{u}{a} + c$$

$$= \sinh^{-1} \frac{x-1}{4} + c$$

Ex 2 $\int \frac{dy}{y^2+10y+30}$

هنا ٣٠ لا يكمل المربع لازم احوره

① معامل $y^2 = 1$

② معامل $y = 10$

③ نصف معامل $y = 5$

مربع $(5)^2 = 25$

$$= \int \frac{dy}{y^2+10y+25-25+30}$$

مربع كامل

$$= \int \frac{dy}{(y+5)^2-25+30}$$

$$= \int \frac{dy}{(y+5)^2+5} \rightarrow$$

هنا نستخدم صيغة $\tan^{-1}$ (الدرس) $= \frac{1}{a}\tan^{-1}\frac{u}{a}+c$

$u = y+5 \qquad dy = dy$

$a^2 = 5$

$a = \sqrt{5}$

$$= \int \frac{du}{u^2+a^2} = \frac{1}{a}\cdot\tan^{-1}\frac{u}{a}+c$$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\frac{y+5}{\sqrt{5}}+c$$

Ex 1 $\int \frac{dx}{4x^2+12x+5}$

لازم معامل $x^2 = 1$ نأخذ عامل مشترك 4

$$= \int \frac{dx}{4\left[(x^2+3x)+\frac{5}{4}\right]}$$

$$= \frac{1}{4}\int \frac{dx}{x^2+3x+\frac{5}{4}}$$

لا يمثل مربع كامل لازم احيد رقم نكمل المربع

$$= \frac{1}{4}\int \frac{dx}{\underbrace{x^2+3x+\frac{9}{4}}_{\text{مربع كامل}}-\frac{9}{4}+\frac{5}{4}}$$

معامل $X = 3$

نصف معامل $X = \frac{3}{2}$

مربع (نصف معامل X) $= \frac{9}{4}$

$$= \frac{1}{4}\int \frac{dx}{(x+\frac{3}{2})^2-\frac{9}{4}+\frac{5}{4}} \longrightarrow -\frac{9}{4}+\frac{5}{4}=\frac{-4}{4}=-1$$

$$= \frac{1}{4}\int \frac{dx}{(x+\frac{3}{2})^2-1}$$

هذه الصيغة هي صيغة $\coth^{-1}$ التي هي

$$\int \frac{du}{a^2-u^2} = -\frac{1}{a}\tanh^{-1}\frac{u}{a}+c$$

$$u^2=(x+\frac{3}{2})^2 \rightarrow u = x+\frac{3}{2}$$

$$du = dx$$

$$a^2=1 \rightarrow a=1$$

$$= \frac{1}{4}\int \frac{du}{u^2-a^2} = \frac{1}{4}\cdot\left(-\frac{1}{a}\coth^{-1}\frac{u}{a}+c\right)$$

$$= -\frac{1}{4}\times\frac{1}{1}\coth^{-1}\frac{x+\frac{3}{2}}{1}$$

$$= -\frac{1}{4}\coth^{-1}\left(x+\frac{3}{2}\right)+c$$

الصيغة العامة لتكامل الدوال العكسية

① $\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + c$

② $\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + c$

③ $\int \frac{du}{a^2-u^2} = \frac{1}{a} \tanh^{-1} \frac{u}{a} + c$

④ $\int \frac{du}{u^2-a^2} = -\frac{1}{a} \coth \frac{u}{a} + c$

⑤ $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a} \operatorname{sech}^{-1} \frac{u}{a} + c$

⑥ $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a} \operatorname{csch}^{-1} \frac{u}{a} + c$

⑦ $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1} \frac{u}{a} + c$

⑧ $\int \frac{du}{u^2+a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c$

EX1 $\int \frac{dx}{\sqrt{-x^2-8x-7}}$

① هذا السؤال لا يمكن حله بالطرق الاعتيادية لان مشتقه داخل الجذر غير متوفرة

② شرط طريقة اكمال المربع ان يكون معامل $x^2 = 1$ وهنا عندي -1 - لازم اسحب السالب خارج القوس

Sol/

$$\int \frac{dx}{\sqrt{-x^2-8x-7}} = \int \frac{dx}{\sqrt{-(x^2+8x+7)}}$$

الرقم 7 لا يكمل المربع

$$= \int \frac{dx}{\sqrt{-(x^2+8x+16-16+7)}}$$

مربع كامل

① معامل $x^2 = 1$

معامل $x = 8$

نصف معامل $x = 4$

مربع معامل $x = 16$

نضيفها 16 ونطرح 16

$$= \int \frac{dx}{\sqrt{-(x+4)^2-9}}$$

$$= \int \frac{dx}{\sqrt{9-(x+4)^2}}$$

رفع السالب

let $a^2 = 9 \Rightarrow a = 3$

$u^2 = (x+4)^2 \Rightarrow u = (x+4)$

$du = dx$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1} \frac{u}{a} + c$$

$$= \sin^{-1} \frac{(x+4)}{3} + c$$

Ex1 $\int \frac{dx}{\sqrt{27-6x-x^2}}$

ملاحظة/

① يجب ان يكون معامل $x^2 = 1$

لازم اغير اماكن الحدود

② معامل $x = 6$

نصف معامل $x = 3$

تربيع معامل $x = (3)^2 = 9$

نضيف ونطرح 9

$$= \int \frac{dx}{\sqrt{-x^2-6x+27}}$$

$$= \int \frac{dx}{\sqrt{-(x^2+6x-27}}$$

لا يكمل المربع

$$= \int \frac{dx}{\sqrt{-(x^2+6x+9-9-27}}$$

$$= \int \frac{dx}{\sqrt{-(x+3)^2-9-27}} = \int \frac{dx}{\sqrt{-(x+3)^2-36}} = \int \frac{dx}{\sqrt{36-(x+3)^2}}$$

let $a^2 = 36 \Rightarrow u^2 = (x+3)^2$ $dx = du$

$a = 6$ $u = (x+3)$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1} \frac{u}{a} + c = \sin^{-1} \frac{x+3}{6} + c$$

lec 4/ Double Integration

$$\iint f(x,y)\,dx\,dy$$

دائماً يكون الدالة بدلالة متغيرين x , y
نكامل الداخلي قبل الخارجي ثم نكامل الخارجي

Ex1 $\int_0^1 \int_1^2 (x+y)\,dx\,dy$

نبدأ نكامل الداخل

$= \int_0^1 \left[\frac{x^2}{2} + yx\right]_1^2 dy$

هنا كاملنا بالنسبة x و y هنا تعتبر ثابت
فقط نعوض لها x من التكامل

$= \int_0^1 \left(\left[\frac{(2)^2}{2} + y*2\right] - \left(\frac{(1)^2}{2} + y*1\right)\right) dy$

$= \int_0^1 \left(2 + 2y - \frac{1}{2} - y\right) dy$

$= \int_0^1 \left(\frac{3}{2} + y\right) dy$

$= \left.\frac{3}{2}y + \frac{y^2}{2}\right]_0^1$

$= \frac{3}{2}(1) + \frac{1}{2}) - (0+0)$

$= \frac{4}{2} = 2$

Ex) Find

$$\int_0^1 \int_x^{x^2} dy\,dx$$

$$= \int_0^1 y \Big|_x^{x^2} .dx$$

$$= \int_0^1 (x^2 - x) . dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 . dx$$

$$= \frac{1}{3} - \frac{1}{2} - 0$$

$$= \frac{-1}{6}$$

Ex)

$$\int_0^2 \int_x^{x^2+1} dy\,dx$$

$$= \int_0^2 y \Big]_x^{x^2+1} dx$$

$$= \int_0^2 ((x^2+1) - x).dx$$

$$= \int_0^2 (x^2 - x + 1) . dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^2$$

$$= \frac{8}{3} - \frac{4}{2} + 2 \Big] - 0$$

$$= \frac{8}{3} - \frac{1}{2} + 2 = \frac{8}{3}$$

H.W /

① $\int_0^1 \int_0^x (x^2 + y)\, dy\, dx$

② $\int_0^2 \int_0^x \left(\frac{x}{1+y}\right) dy\, dx$

③ $\int_0^1 \int_0^x (e^x + y^2)\, dy\, dx$

Area by Double Integration

$$A = \iint_R dA \begin{cases} \to dx\, dy \\ \to dy\, dx \end{cases}$$

خطوات الحل

① رسم المنطقة

② ايجاد نقاط التقاطع

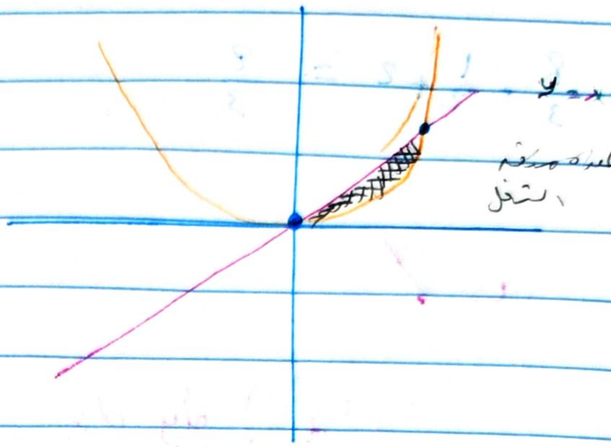
③ اخذ شريحة (افقية او عمودية)!!

Ex! Find the area of the region bounded by $y = x$ and $y = x^2$ in the first quadrant. يريد المساحة بالربع الأول

① نرسم الدوال

الدالة الأولى $x = y$

0	0
1	1
-1	-1
2	2



$y = x^2$

x	y
-1	1
0	0
1	1
-2	4

② نعين نقاط التقاطع

حتى نعين نقاط التقاطع نساوي الدالتين

$y = y$

$x = x^2 \Rightarrow x - x^2 = 0$

$x(1 - x) = 0$

$x = 0$ or $x = 1$

نعوض بأي معادلة حتى نعرف قيم ال y ولنأخذ معادلة

$y = x$

$y = 0$ ∴ النقطة الأولى $(0,0)$

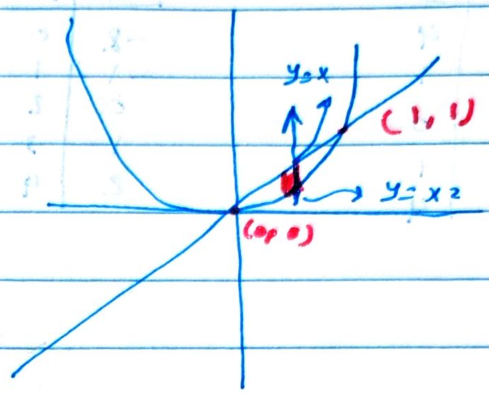
عندما $x = 1$ نعوض بأي معادلة لنعرف قيم y

$x = y$

$y = 1$ النقطة $(1,1)$

③ نحدد الشريحه اما عموديه او افقيه

الشريحه اما تكون عموديه موازيه لمحور y او افقيه موازيه لمحور X



$$A = \int_0^1 \int_{x^2}^{x} 1 \, dy \, dx$$

دالة داله ... الداله هي ١

① نخرج حدود التكامل الخارجي يكون عدد ثابت (ارقام) يعني بصيغة الحالة استخرج حدود X لان حدود التكامل dx فحدود التكامل تكون $\int_0^1$ اي ابتدينا من ٠ وانتهينا ب ١

② اما الداخلي فهو شغل دوال يعني الداله مثلا بدينا من اي داله هي داله $y = x^2$ وانتهى بالداله $y = x$ اذا الحد الادنى هو x^2 والاعلى هو X

$$A = \int_0^1 \int_{x^2}^{x} dy \, dx$$

$$= \int_0^1 y \Big]_{x^2}^{x} \cdot dx$$

$$= \int_0^1 (x - x^2) \cdot dx = ~~\frac{x^3}{3} - \frac{x^2}{2} \Big]_0^1~~ =$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Ex.1 The parabola $y = x^2$ and the line $y = x + 2$

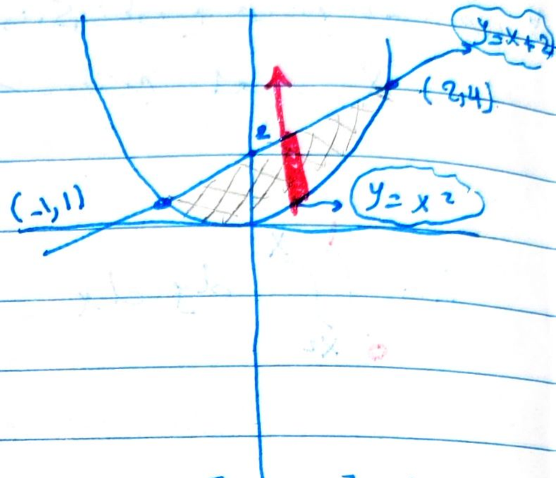
$y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

$y = x + 2$

x	y
-2	0
-1	1
0	2
1	3
2	4

① نرسم الداله



② نجد نقاط التقاطع

نحل بالتجربه

$y = y \rightarrow x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

يعني اما $x - 2 = 0$

$x = 2$

او $x + 1 = 0$

$x = -1$

اما

$x = 2$ or $x = -1$

عشان نعوضهم بالداله: $y = 4$, $y = 1$

$(2, 4)$, $(-1, 1)$

③ نكتب الداله التكامل وافد شريحه (عموديه)

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx$$

حدود x راح يكون من -1 الى 2

هي قيم x من نقاط التقاطع

اما dy من اي داله بدء الشريحه هو

$y = x^2$ وينتهي بداله $y = x + 2$

اذا الحدود يكون الحد الادنى x^2 وينتهي بـ $x + 2$

$$= \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$$

$$= \int_{-1}^{2} y \Big]_{x^2}^{x+2} \cdot dx$$

$$= \int_{-1}^{2} (x+2-x^2] \cdot dx$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big]_{-1}^{2} = \frac{4}{2} + 4 - \frac{8}{3}\Big] - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= \frac{9}{2}$$

H.W/

① Find the area bounded between Two Curves $y = x^2$ and $y = 2x$ by using double integration

② Find the area of the region bounded by two curves $y = \sqrt{x}$ and $y = x$ by using double integration

lec 6 / Triple Integration

مثل التكامل الثنائي نبدأ نكامل
من الداخل للخارج يعني اول شي
اكامل dz

Ex)

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\, dy\, dx$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} z \Big]_0^{\sqrt{9-x^2}} dy\, dx$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-x^2}\, .dy\, dx$$

$$= \int_0^3 \sqrt{9-x^2}\, .\, y \Big]_0^{\sqrt{9-x^2}} dx$$

$$= \int_0^3 \sqrt{9-x^2} . \sqrt{9-x^2}\, .dx$$

$$= \int_0^3 (9-x^2)\, .dx$$

$$= 9x - \frac{x^3}{3} \Big|_0^3 = 27 - \frac{27}{3} = 18$$

Ex)

$$\int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^1 xyz\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 xy \frac{z^2}{2} \Big|_0^1 dy\, dx$$

$$= \int_0^1 \int_0^1 xy \cdot \frac{1}{2} \cdot dy\,dx$$

$$\int_0^1 \frac{x}{2} \frac{y^2}{2} \Big|_0^1 \, dx$$

$$= \int_0^1 \frac{x}{2} \cdot \frac{1}{2} \cdot dx = \int_0^1 \frac{x}{4} \cdot dx$$

$$= \frac{x^2}{2} \cdot \frac{1}{4} \Big|_0^1 = \frac{1}{8}$$

Triple Integration Methods:-

① Rectangular (Cartesian) (x, y, z)

هنا يجوز التبديل بيهم

يعني $dx\,dy\,dz$ او $dy\,dx\,dz$, $dz\,dy\,dx$

$$dV = dx\,dy\,dz$$

② Cylindrical (r, θ, z)

الترتيب اجباري لايمكن تغيير الحدود

$$dV = r\,dz\,dr\,d\theta$$

③ Spherical (ρ, θ, ϕ)

هنا يمكن التبديل فقط بين

$d\theta$ و $d\phi$

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Ex) Rectangular (cartesian) (x, y, z)

Ex) Find the Volume of the Solid bounded by the planes (x=1), (x=3), y=2, y=3 and z=2 and z=4.

$$V = \int_2^4\int_2^3\int_1^3 dx\,dy\,dz$$

$$= \int_2^4\int_2^3 x \Big|_1^3 dy\,dz$$

$$= \int_2^4\int_2^3 (3-1)\cdot dy\,dz = \int_2^4\int_2^3 2\cdot dy\,dz$$

$$= \int_2^4 2y \Big|_2^3 dz = \int_2^3 2(3-2)\cdot dz = \int_2^3 2\cdot dz$$

$$= 2z \Big|_2^4 = 4 \text{ unit}^3$$

* ~ * ~ * ~ *

Ex) Find the volume of the region bounded by $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$

$$V = \int_0^1\int_0^2\int_0^3 dz\,dy\,dx$$

$$V = \int_0^1\int_0^2 z \Big|_0^3 dy\,dx = \int_0^1\int_0^2 3\cdot dy\,dx$$

$$= \int_0^1 3y \Big|_0^2 \, dx$$

$$= \int_0^1 3 * (2-0) \, . dx = \int_0^1 6 . dx$$

$$= 6 * x \Big|_0^1 = \underline{\underline{6}} \text{ unit}^3$$

* * * * *

(2) Cylinderical coordinat

$$r^2 = x^2 + y^2$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

} cylinderal coordinat.

Area $A = \int r \, dr \, d\theta$

هي امتداد للاحداثيات القطبية في مستوى xy مع اضافة الارتفاع Z عامود

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$$dv = r \, dz \, d\theta \, d\theta$$

Ex) Find the volume of the cylinder $r \leq 2$ $0 \leq z \leq 5$

$0 \leq \theta \leq 2\pi$

حدود كاملة

$$dv = \int_0^{2\pi}\int_0^{2}\int_0^{5} r \cdot dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r \cdot z\Big|_0^5 dr \cdot d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 5r \cdot dr \cdot d\theta$$

$$= \int_0^{2\pi} \frac{5r^2}{2}\Big|_0^2 d\theta$$

$$= \frac{5}{2}\int_0^{2\pi} (4-0)\, d\theta$$

$$= \int_0^{2\pi} 10 \cdot d\theta = 10\,\theta\Big|_0^{2\pi}$$

$$V = 20\pi$$

Method Of Integration

1. Integration By Part

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow u\frac{dv}{dx} = \frac{d}{dx}uv - v\frac{du}{dx}$$

$$\Rightarrow \int u\frac{dv}{dx} = \int \frac{d}{dx}uv - \int v\frac{du}{dx}$$

$$\Rightarrow \int u\frac{dv}{dx} = uv - \int v\frac{du}{dx} \quad \ldots\ldots *$$

Then equation (*) is the integration by part formula. In this method the term $\int udv$ is converted to another form that is integrated easily.

Ex: By using integration by part formula find the integral of $x\, cosx\, dx$.

Sol:

$$\int x \cos x \, dx.$$

We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Ex: By using integration by part formula find the integral of $ln\, x\, dx$.

Sol: Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $v = x.$ Simplest antiderivative

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

Ex : Find the integral of $x^2 e^x dx$

Sol: $$\int x^2 e^x \, dx.$$

With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Hence,

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx$$

$$= x^2 e^x - 2x e^x + 2e^x + C.$$

Ex : Find the integral of $cosx \; e^x dx$

$$\int e^x \cos x \, dx.$$

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Home work:

1. $\int x \sin \frac{x}{2} \, dx$
2. $\int \theta \cos \pi\theta \, d\theta$
3. $\int t^2 \cos t \, dt$
4. $\int x^2 \sin x \, dx$
5. $\int_1^2 x \ln x \, dx$
6. $\int_1^e x^3 \ln x \, dx$
7. $\int \tan^{-1} y \, dy$
8. $\int \sin^{-1} y \, dy$
9. $\int x \sec^2 x \, dx$
10. $\int 4x \sec^2 2x \, dx$
11. $\int x^3 e^x \, dx$
12. $\int p^4 e^{-p} \, dp$

2. An Integral with an even power of $sin\ x$ and $cos\ x$:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

For odd power. $sin^2x = 1 - cos^2x$ and $cos^2x = 1 - sin^2x$.

3. Integration of the products of sines and cosines.

$\int sin^m x\ cos^n x$	Relevant Identities
n odd	$cos^2x = 1 - sin^2x.$
m even	$sin^2x = 1 - cos^2x$
m even n odd	$sin^2x = \frac{1}{2}(1 - cos2x)$ $cos^2x = \frac{1}{2}(1 + cos2x)$

4. Integration of the products of $sinmx\ cosnx$

$$\int \sin mx \cos nx\, dx, \quad \int \sin mx \sin nx\, dx, \quad \int \cos mx \cos nx\, dx$$

can be found by using the trigonometric identities

$$\sin\alpha \cos\beta = \tfrac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$\sin\alpha \sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha \cos\beta = \tfrac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

5. Partial Fractions

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$,
	where x^2+bx+c cannot be factorised further	

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\cdot \ln|ax+b| + C$$

$$\int \frac{1}{(ax+b)^2}dx = -\frac{1}{a}\cdot\frac{1}{ax+b} + C$$

$$\int \frac{x}{ax^2+b}dx = \frac{1}{2a}\cdot \ln|ax^2+b| + C$$

$$\int \frac{1}{ax^2+1}dx = \frac{1}{\sqrt{a}}\cdot arctan(\sqrt{a}\,x) + C$$

Ex: Evaluate the integral of $\frac{x+5}{(x-4)(x-1)}$.

Sol: $\frac{x+5}{(x-4)(x-1)} = \frac{A}{x-4} - \frac{B}{x-1}$

$$x+5 = A(x-1) + B(x-4)$$

$$1 = A + B$$

$$5 = -A - 4B$$

$$6 = -3B \text{ then } B = -2,\ A = 3$$

$$\frac{x+5}{(x-4)(x-1)} = \frac{3}{x-4} + \frac{2}{x-1}$$

Then $\int \frac{x+5}{(x-4)(x-1)}\, dx = \int \left(\frac{3}{x-4} + \frac{2}{x-1}\right) dx = 3ln\,(x-4) + 2ln(x-1) + c$

Ex: Evaluate the integral of $\frac{2x-2}{(x+5)(x+2)(x-3)}$

Sol:

$$\frac{2x-2}{(x+5)(x+2)(x-3)} = \frac{A}{x+5} + \frac{B}{x+2} + \frac{C}{x-3} .$$

$$2x-2 = A(x+2)(x-3) + B(x+5)(x-3) + C(x+5)(x+2) .$$

Substitute $x=-2$. $-4-2 = 0 + B(3)(-5) + 0 \rightarrow -6 = -15B \rightarrow B = \frac{2}{5}$

Substitute $x=3$. $6-2 = 0+0+C(8)(5) \rightarrow 4 = 40C \rightarrow C = \frac{1}{10}$

Substitute $x=-5$. $-12 = A(-3)(-8)+0+0 \rightarrow -12 = 24A \rightarrow A = -\frac{1}{2}$

The partial fraction decomposition is $\frac{2x-2}{(x+5)(x+2)(x-3)} = -\frac{1}{2}\cdot\frac{1}{x+5} + \frac{2}{5}\cdot\frac{1}{x+2} + \frac{1}{10}\cdot\frac{1}{x-3}$.

$$\int \frac{2x-2}{(x+5)(x+2)(x-3)}\, dx = \int \left[-\frac{1}{2}\frac{1}{x+5} + \frac{2}{5}\frac{1}{x+5} + \frac{1}{10}\frac{1}{x-3}\right] dx$$

$$= -\frac{1}{2}\ln(x+5) + \frac{2}{5}\ln(x+5) + \frac{1}{10}\ln(x-3) + c$$

Ex: Evaluate the integral of $\frac{4x}{(x-2)^2}$

Sol:

$$\frac{4x}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} .$$

$$4x = A(x-2) + B .$$

$\boldsymbol{A = 4}, 0 = -2 * 4 + B$ then $\boldsymbol{B = 8}$

$$\int \frac{4x}{(x-2)^2}\, dx = \int \left[\frac{4}{x-2} + \frac{8}{(x-2)^2}\right] dx = 4\,ln\,(x-2) - \frac{8}{x-2} + c$$

Ex: Evaluate the integral of $\frac{x^2-2x-5}{x^3-5x^2}$

$$\frac{x^2-2x-5}{x^3-5x^2} = \frac{x^2-2x-5}{x^2(x-5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-5} .$$

Multiply by $x^2(x-5)$ to get $x^2-2x-5 = Ax(x-5) + B(x-5) + Cx^2$.

Substitute $x=0$. $-5 = 0 + B(-5) + 0 \rightarrow B = 1$

Substitute $x=5$. $10 = 0 + 0 + C(25) \rightarrow C = \frac{2}{5}$

Substitute $x=1$. $-6 = A(-4) + (-4) + \frac{2}{5}(1) \rightarrow A = \frac{3}{5}$

$$\int \frac{x^2-2x-5}{x^3-5x^2} = \int \left[\frac{3/5}{x} + \frac{1}{x^2} + \frac{2/5}{x-5}\right] dx$$

$$= \frac{3}{5} lnx - \frac{1}{x} + \frac{2}{5}\ln(x-5) + c$$

Ex: Find the integral of $\frac{x^2+4x+12}{(x-2)(x^2+4)}$

$$\frac{x^2+4x+12}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4} .$$

Multiply by $(x-2)(x^2+4)$ to get $x^2+4x+12 = A(x^2+4) + (Bx+C)(x-2)$.

Substitute $x=2$. $24 = A(8) + 0 \rightarrow A = 3$

Substitute $x=0$. $12 = 3(4) + (C)(-2) \rightarrow C = 0$

Substitute $x=1$. $17 = 3(5) + (B)(-1) \rightarrow B = -2$

$$\int \frac{x^2+4x+12}{(x-2)(x^2+4)} dx =$$

$$\int \left[\frac{3}{(x-2)} + \frac{-2x}{(x^2+4)}\right] dx = 3\ln(x-2) - \ln(x^2+4) + C$$

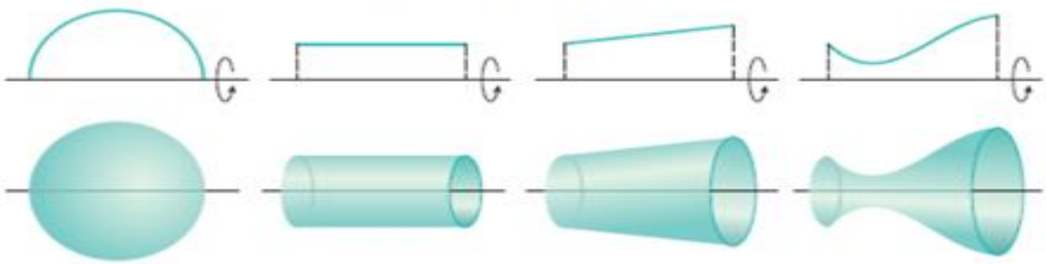
Home work:

1. $\int \frac{3-4x}{x^2+x}\,dx$
2. $\int \frac{x}{x^2+7+10}\,dx$
3. $\int \frac{6}{3x^2-14x+8}\,dx$
4. $\int \frac{3x^2+8x-7}{(x+4)(x+3)(x+1)}\,dx$
5. $\int \frac{2-4x^2}{(x+2)(x-2)(x-5)}\,dx$
6. $\int \frac{3x}{(x+4)(x-1)(x-3)}\,dx$
7. $\int \frac{3-2x}{x^2+6x+9}\,dx$
8. $\int \frac{3x-1}{x^3-2x^2}\,dx$
9. $\int \frac{2x^2+x+4}{(x+1)(x-4)^2}\,dx$

Areas of surface of revolution

A surface of revolution is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve.

Some Surfaces of Revolution



A) The area of the surface (S) generated by revolving the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis.

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

B) The area of the surface (S) generated by revolving the curve $x = g(y)$ between $y = c$ and $y = d$ about the y-axis.

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\,dy = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Example(1)

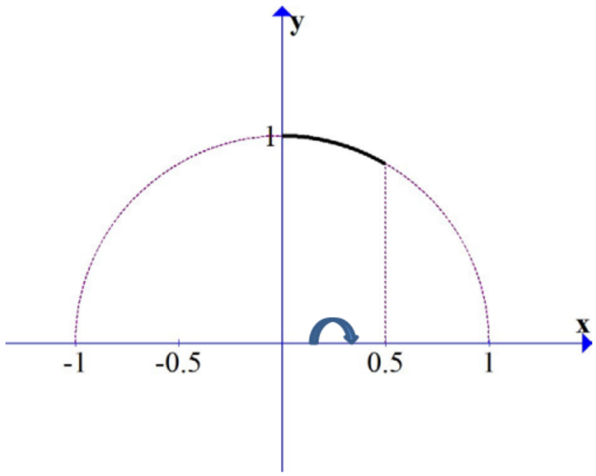
Find the surface area generated by revolving the curve

$$y = \sqrt{1 - x^2}, \qquad 0 \le x \le \frac{1}{2}$$

about the x-axis.

Solution:

The graph of the curve is the upper semi-circle of radius 1 centered at the origin.



$$y = \sqrt{1-x^2}$$
$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$
$$= \int_0^{1/2} 2\pi\sqrt{1-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx$$
$$= \int_0^{1/2} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx$$
$$= \int_0^{1/2} 2\pi\sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx$$
$$= \int_0^{1/2} 2\pi\,dx$$
$$= \cdots = \pi$$

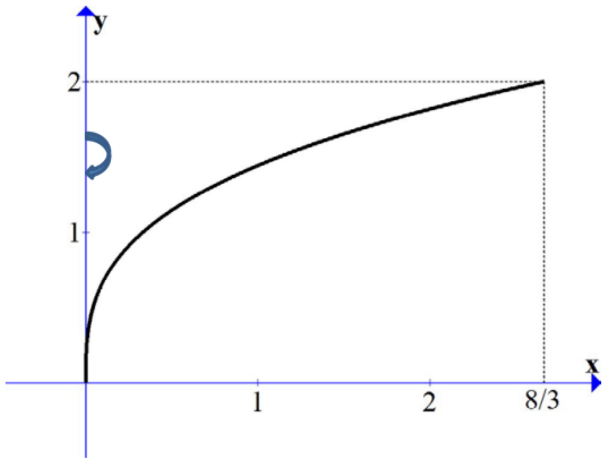
Find the surface area generated by revolving the curve

$$y = \sqrt[3]{3x}, \qquad 0 \le y \le 2$$

about the y-axis.

<u>Solution</u>:

$$y = \sqrt[3]{3x} \Rightarrow x = \frac{1}{3}y^3$$
$$\frac{dx}{dy} = y^2$$



$$S = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy$$
$$= \int_0^2 2\pi \left(\frac{1}{3}y^3\right)\sqrt{1 + (y^2)^2}\, dy$$
$$= \int_0^2 \frac{2\pi}{3} y^3 \sqrt{1 + y^4}\, dy$$

$$u = 1 + y^4$$
$$du = 4y^3 dy \Rightarrow \frac{1}{4}du = y^3 dy$$
$$y = 2 \Rightarrow u = 17$$
$$y = 0 \Rightarrow u = 1$$

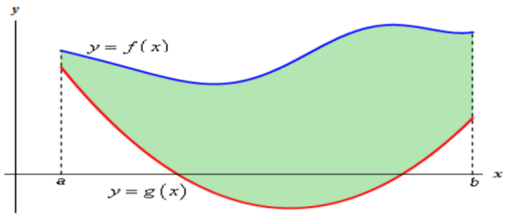
$$S = \int_0^2 \frac{2\pi}{3} y^3 \sqrt{1 + y^4}\, dy$$
$$= \int_1^{17} \frac{2\pi}{3}\sqrt{u} \cdot \frac{1}{4} du$$
$$= \cdots = \frac{\pi}{9}\left(17\sqrt{17} - 1\right)$$

Application of integration

1. Area Between Two Curves

A- Integration with respect to x

In the first case the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ is determined. If it is assumed that $f(x) \geq g(x)$.

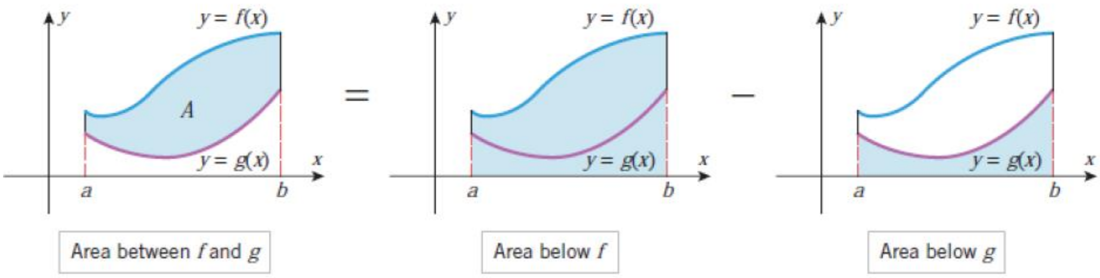


$$A = \int_a^b f(x) - g(x)\,dx$$

B- Integration with respect to y

The area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$ is given by :

$$A = \int_c^d f(y) - g(y)\,dy$$



Ex: Find the area bounded by the x-axis and the curve : $y = 2x - x^2$.

Sol: firstly the point of intersection between the curve and x- axis is calculated by making $y = 0$,

$$2x - x^2 = 0$$

$x(2 - x) = 0$. *Either* $x = 0$ *or* $x = 2$, $A = \int_0^2 (2x - x^2)\,dx =$

$$\left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = 1.33 \text{ squared unit}$$

Ex: Find the area bounded by the y-axis and the curve $x = y^2 - y^3$

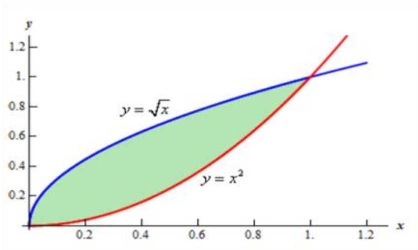
Sol: For $x = 0$ then $y^2(1 - y) = 0$, $y = 0$ and $y = 1$

$$A = \int_0^1 (y^2 - y^3)\ dy = \frac{y^3}{3} - \frac{y^4}{4}\Big]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Ex: Determine the area of the region enclosed by: $y = x^2$ and $y = \sqrt{x}$

Sol: For $y = \sqrt{x} - x^2 = 0$, $\sqrt{x}\left(1 - x^{3/2}\right) = 0$, $x = 0$ and $x = 1$

$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \left[\frac{2x^{3/2}}{3} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$



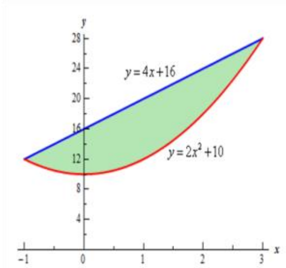
Ex: Determine the area of the region bounded by: $y = 2x^2 + 10$ and $y = 4x + 16$

Sol: For $y = 4x + 16 - 2x^2 - 10 = -2x^2 + 4x + 6$

$y = 0, -2x^2 + 4x + 6 = 0$

$x^2 - 2x - 3 = (x - 3)\ (x + 1)$

$x = 3$ and $x = -1$



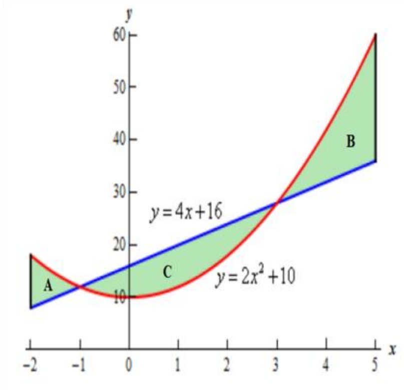
$$A = \int_{-1}^{3} [-2x^2 + 4x + 6]\, dx = -\frac{2x^3}{3} + 2x^2 + 6x\Big]_{-1}^{3}$$

$$= \left[\frac{-2 * 27}{3} + 2 * 9 + 6 * 3 - \left(\frac{+2}{3} + 2 - 6\right)\right]$$

$= -18 + 18 + 18 - (-3.334) = 21.334$

Ex: Determine the area of the region bounded by: $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$, and$x = 5$

Sol: The area from $x = -1$ to $x = 3$ is (area C) calculated from the above example. Areas (A&B) are calculated by:



$$A_{A\&B} = \int_{-2}^{-1} (2x^2 + 10 - 4x - 16)dx + \int_{3}^{5} (2x^2 + 10 - 4x - 16)dx$$

$$= \int_{-2}^{-1} (2x^2 - 4x - 6)dx + \int_{3}^{5} (2x^2 - 4x - 6)dx$$

$$= \frac{2x^3}{3} - 2x^2 - 6x\Big]_{-2}^{-1} + \frac{2x^3}{3} - 2x^2 - 6x\Big]_{3}^{5}$$

$$= \frac{2(-1+8)}{3} - 2(1-4) - 6(-1+2) + 2\frac{125-27}{3} - 2(25-9) - 6(5-3)$$

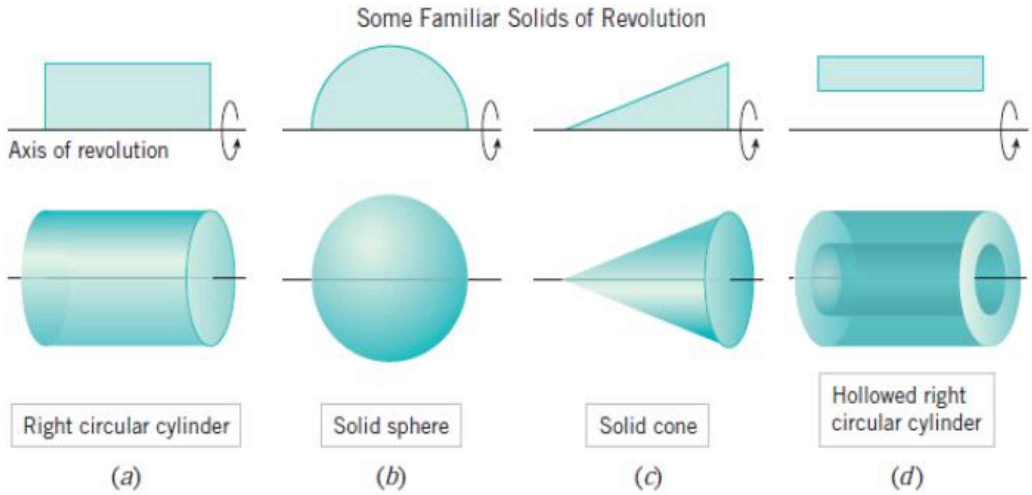
$$= \frac{14}{3} + 6 - 6 + 2\frac{98}{3} - 2(16) - 12 = \frac{14}{3} + \frac{196}{3} - 32 - 12$$

$$= \frac{210}{3} - 44 = 70 - 44 = 26$$

The total area is $A_{A\&B} + A_C = 26 + 21.3 = 47.3$

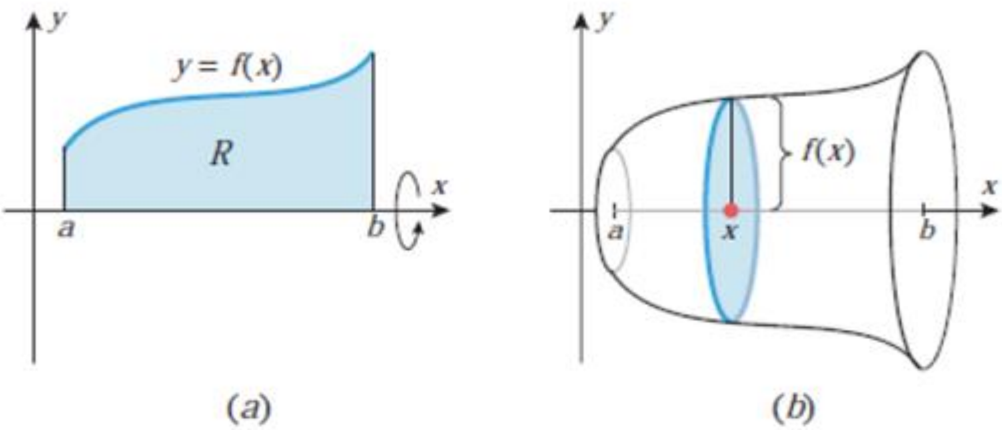
2. Volume of the Solids of revolution

A Solid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the **axis of revolution.**



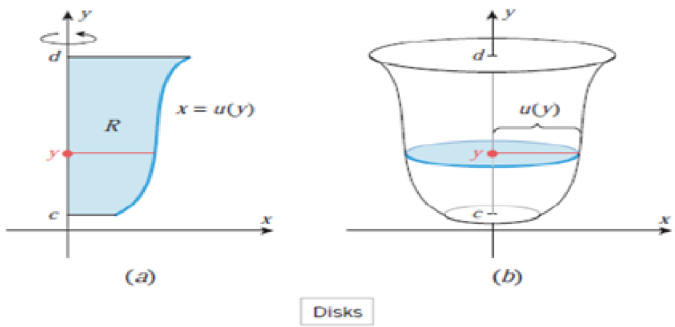
The Disk method

A)Revolution about x- axis



Revolution about x-axis

B) Revolution about Y- axis



Revolution about Y -axis

$V=\int_a^b A(x)dx$	$V=\int_c^d A(y)dy$

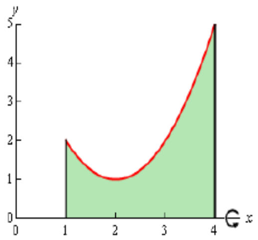
$A=\pi(\text{radius})^2$ A= $\pi(R_{(x)})^2$ A= $\pi(R_{(y)})^2$

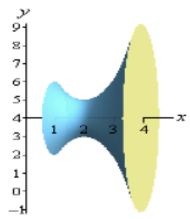
$$A(x)=\pi[f(x)]^2$$

$$V=\int_a^b \pi[f(x)]^2\,dx \qquad V=\int_c^d \pi[u(y)]^2\,dy$$

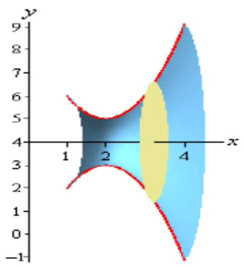
Ex: Determine the volume of the solid generated by rotating the region bounded by $f(x)=x^2-4x+5$ and $x=1, x=4$ about the x-axis.

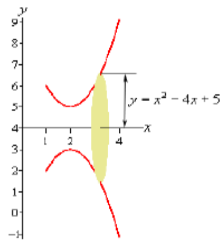
Step 1 is to sketch the bounding region and the solid obtained by rotating the region about the x-axis. Here are both of these sketches.





Step 2: To get a cross section we cut the solid at any x, since the x-axis it the axis of rotation.





$$A(x) = \pi r^2 = \pi[f(x)]^2$$

$$A(x) = \pi(x^2 - 4x + 5)^2 = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25)$$

Step3. Determine the boundaries which will represent the limits of integration. Working from left to right the first cross section will occur at $x = 1$, and the last cross section will occur at $x = 4$. These are the limits of integration.

Step 4. Integrate to find the volume:

$$V = \int_a^b A(x)dx = \pi\int_a^b f(x)dx = \pi\int_1^4 (x^2 - 4x + 5)^2 dx = \pi\int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25)dx =$$

$$\pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Bigg|_1^4 = \frac{78\pi}{5}$$

2. Finding volume of a solid of revolution using a washer method.

This is an extension of the disc method. The procedure is essentially the same, but now we are dealing with a hollowed object and two functions instead of one, so we have to take the difference of these functions into the account.

The general formula in this case would be:

$A = \pi(R^2 - r^2)$ where R is an outer radius and r is the inner radius.

FORMULAS: V= $\int A(x)dx$, or respectively $\int A(y)dy$

1. The volume of the solid generated by a region between *f(x)* and *g(x)* bounded by the vertical lines x=a and x=b, which is revolved about the x-axis is

$$V = \pi\int_a^b \left|(f(x))^2 - (g(x))^2\right|dx \qquad \text{(washer with respect to x)}$$

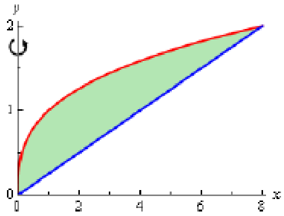
2. The volume of the solid generated by a region between *f(y)* and *g(y)* bounded by the horizontal lines y=c and y=d which is revolved about the y-axis.

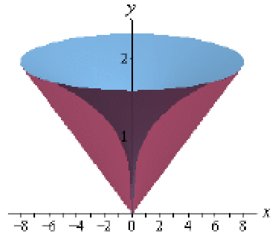
$$V = \pi\int_c^d \left|(f(y))^2 - (g(y))^2\right|dy \qquad \text{(washer with respect to y)}$$

Ex: Determine the volume of the solid generated by rotating the region bounded by $y = \sqrt[3]{x}$, and $y = \frac{x}{4}$ that lies in the first quadrant about the y-axis.

Solution

Step 1: Graph the bounding region and a graph of the object. The cross section is cut **perpendicular** to the axis of rotation and it is a horizontal washer. The inner and outer radii of the washer are x values, so we will need to rewrite our functions into the form $x = f(y)$.

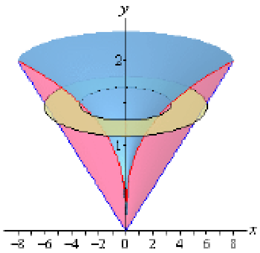


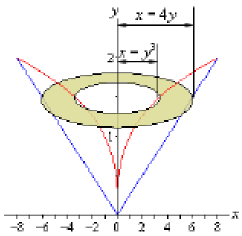


Here are the functions written in the correct form for this example.

$$y=\sqrt[3]{x}\Rightarrow x=y^3 \quad \text{and} \quad y=\frac{x}{4}\Rightarrow x=4y$$

Step 2. Graph couple of sketches of the boundaries of the walls of this object as well as a typical washer. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.





The cross-sectional area is then, $A(y)=\pi\left((4y)^2-\left(y^3\right)^2\right)=\pi\left(16y^2-y^6\right)$

Step 3. Working from the bottom of the solid to the top we can see that the first cross-section will occur at $y=0$ and the last cross-section will occur at $y=2$. These will be the limits of integration.

Step 4. The volume is then, $V=\int_c^d A(y)dy=\pi\int_0^2\left(16y^2-y^6\right)dy=\pi\left(\frac{16}{3}y^3-\frac{1}{7}y^7\right)\Big|_0^2=\frac{512\pi}{21}$

Lecture ٣ : Vectors

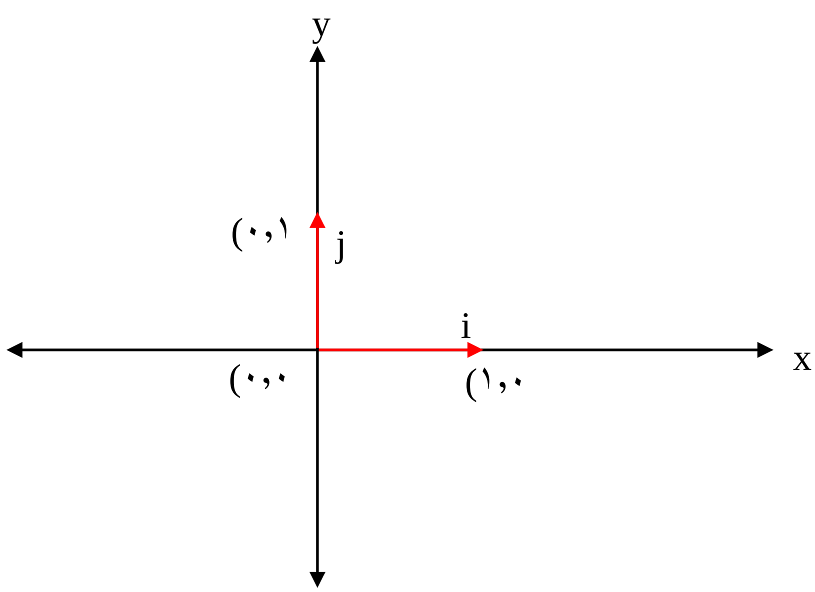
Vector components:

Quantities can be divided into:

١) scalar: which is defined by magnitude only, for example mass, time...

٢) vector: the quantity which needs magnitude and direction, for example force, velocity, acceleration....

We shall denote the vector from (٠,٠) to (١,٠) by (i) and the vector from (٠,٠) to (٠,١) by (j) as shown in figure below:



Then any vector in the x - y plane can be divided in the terms of i and j.

Ex. Sketch the following vectors:

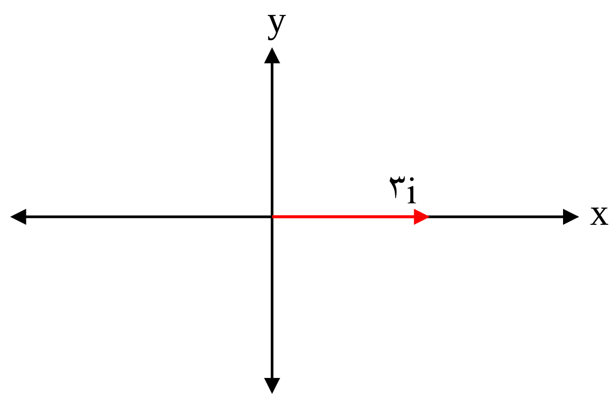
a) ٣i

b) - ٤i

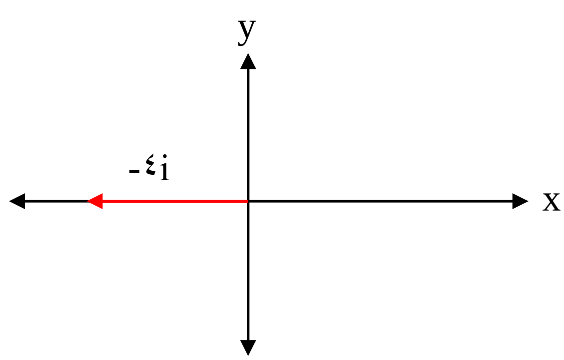
c) -٢i + ٣j

Sol.:

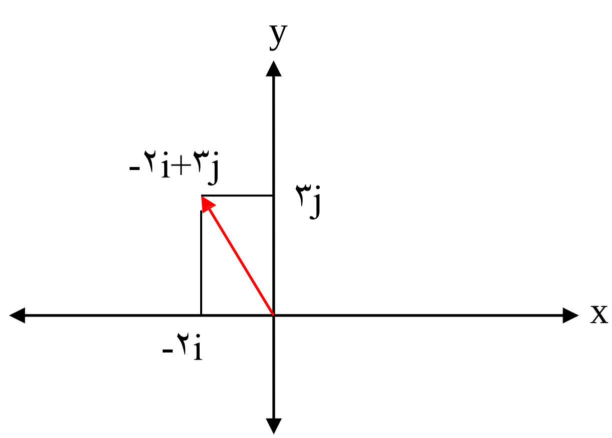
a)



b)



c)



Note: when we say the vector $\vec{AB}$ means: This vector directed from point A to point B.

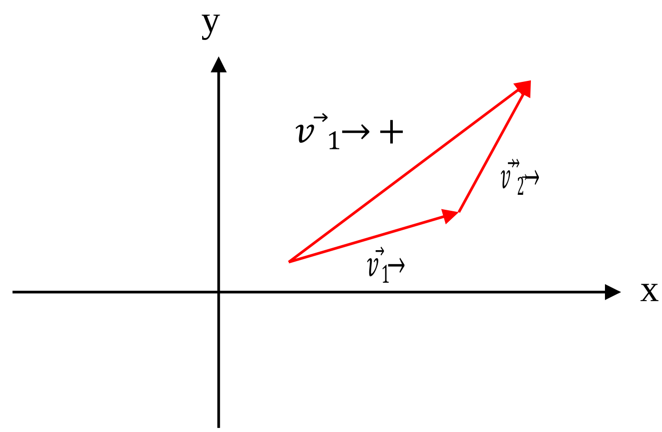
Arithmetic operation on vector:

١) Addition:

Let $\vec{v}_1 = a_1 i + b_1 j$ and $\vec{v}_2 = a_2 i + b_2 j$

Then $\vec{v}_1 + \vec{v}_2 = a_1 i + b_1 j + a_2 i + b_2 j$

$$= (a_1 + a_2)i + (b_1 + b_2)j$$

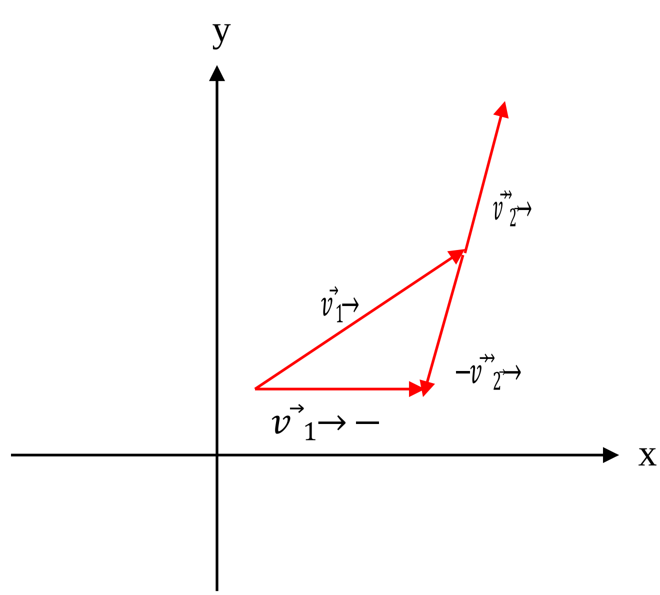


٢) Subtraction:

Let $\vec{v}_1 = a_1 i + b_1 j$ and $\vec{v}_2 = a_2 i + b_2 j$

Then $\vec{v}_1 - \vec{v}_2 = (a_1 i + b_1 j) - (a_2 i + b_2 j)$

$$= (a_1 - a_2)i + (b_1 - b_2)j$$



٣) Multiplication of vector by scalar:

Let $\vec{v} = ai + bj$ and c is scalar.

then $c\,\vec{v} = c(ai + bj) = cai + cbj$

Definitions:

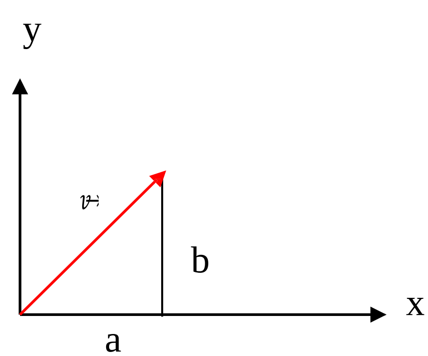
١) Length of vector:

The length of vector $\vec{v} = ai + bj$ is usually denoted by $|\vec{v}|$ which may be read as "The magnitude of $\vec{v}$" .

$|\vec{v}|$ is found using Pythagoras theorem and as follows:

$\vec{v} = ai + bj$

$|\vec{v}| = \sqrt{a^2 + b^2}$



٢) Zero vector:

Any vector of length zero is called zero vector $\vec{0}$,

$ai + bj = 0$

$ai + bj = 0i + 0j$ if and only if a = b = ٠.

٣) Unit vector:

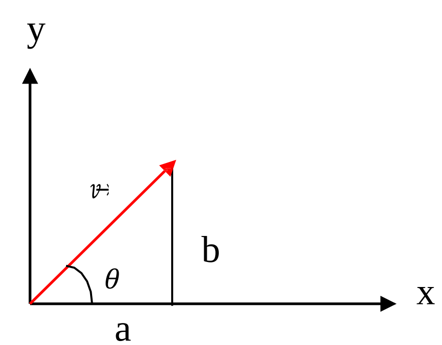
It is part from any vector. This part has length equal to unity and it is used to describe the direction of the vector.

$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ where $\vec{u}$ is unit vector of $\vec{v}$.

Ex. Find the unit vector of $\vec{v} = ai + bj$ and prove that unit vector depends on the angle between the vector and the x - axis?

Sol.:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$



$$\vec{u} = \frac{ai + bj}{\sqrt{a^2+b^2}}$$

$$\vec{u} = \frac{a}{\sqrt{a^2+b^2}} i + \frac{b}{\sqrt{a^2+b^2}} j$$

$$\vec{u} = \cos\theta i + \sin\theta j$$

Ex. Find the unit vector of $\vec{A} = 3i + 4j$.

Sol.:

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|}$$

$$|\vec{A}| = \sqrt{3^2 + 4^2} = ٥$$

$$\vec{u} = \frac{3i+4j}{٥} = ٠,٦ i + ٠,٨ j$$

since

$$\vec{u} = \cos\theta i + \sin\theta j$$

$$\cos\theta = ٠,٦$$

$$\therefore \quad \theta = ٥٣^{o}$$

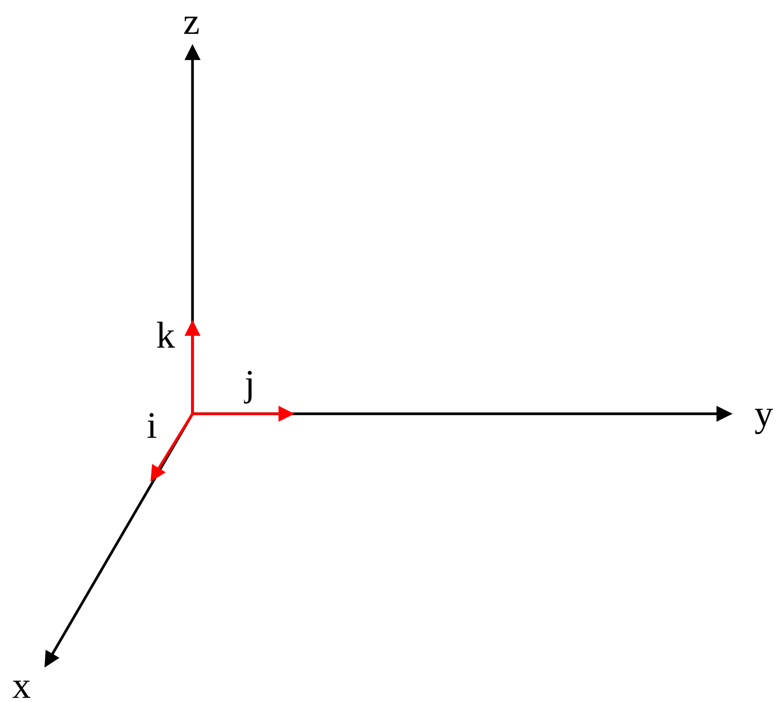
Vector in Space:

Now we shall consider the vector in three dimensional space as follows:

(i) as a vector pointing from (٠,٠,٠) to (١,٠,٠)

(j) as a vector pointing from (٠,٠,٠) to (٠,١,٠)

(k) as a vector pointing from (٠,٠,٠) to (٠,٠,١)



Any vector $\vec{A}$ for example may be represented as:

$$\vec{A} = ai + bj + ck$$

and

$$|\vec{A}| = \sqrt{a^2 + b^2 + c^2}$$

Ex.: Find a unit vector in the direction of vector from p١(١,٠,١) and p٢(٣,٢,٠).

Sol.:

$\overrightarrow{p_1 p_2}$ = (٣ - ١)i + (٢ - ٠)j + (٠ - ١)k

$\overrightarrow{p_1 p_2}$ = ٢i + ٢j - k

$|\overrightarrow{p_1 p_2}| = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$

$\vec{u} = \frac{\overrightarrow{p_1 p_2}}{|\overrightarrow{p_1 p_2}|} = \frac{2i+2j-k}{٣} = \frac{2}{٣} i + \frac{2}{٣} j - \frac{1}{٣} k$

Scalar product (dot product):

The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$, denoted by $\vec{A}.\vec{B}$ (read $\vec{A}$ dot $\vec{B}$), is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle θ between them.

$\vec{A}.\vec{B} = |\vec{A}|.|\vec{B}|\ cos\theta$, ٠ $\leq \theta \leq \pi$

The following laws was valid:

١. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

٢. $\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

٣. i.i = j.j=k.k = ١, i.j = j.k = k.i = ٠

The dot product can be used to find:

١) The angle between two vectors.

٢) The projection of vector $\vec{B}$ on $\vec{A}$ = $|\vec{B}|\ cos\theta = \frac{\vec{A}\vec{B}}{|A|}$

٣) The projection of $\vec{A}$ on $\vec{B}$ = $|\vec{A}|\ cos\theta = \frac{\vec{A}\vec{B}}{|B|}$

Ex. Find the angle between $\vec{A} = i - 2j - 2k$ and $\vec{B} = 6i + 3j + 2k$ also find the projection of $\vec{A}$ on $\vec{B}$ and $\vec{B}$ on $\vec{A}$.

Sol.:

١) $\vec{A}.\vec{B} = |\vec{A}|.|\vec{B}| \cos\theta$

$$\cos\theta = \frac{\vec{A}\vec{B}}{|\vec{A}|.|\vec{B}|}$$

$\vec{A}.\vec{B}$ = ١*٦ + (-٢)*٣ + (-٢)*٢ = - ٤

$$|\vec{A}| = \sqrt{1+4+4} = 3$$

$$|\vec{B}| = \sqrt{36+9+4} = 7$$

$$\cos\theta = \frac{-4}{٣*٧} = \frac{-4}{٢١}$$

$$\theta = 101^o$$

The projection of $\vec{A}$ on $\vec{B}$ $= |\vec{A}|\, cos\theta = \frac{\vec{A}.\vec{B}}{|B|} = \frac{-4}{٧}$

٢) The projection of vector $\vec{B}$ on $\vec{A}$ $= |\vec{B}|\ cos\theta = \frac{\vec{A}.\vec{B}}{|A|} = \frac{-4}{٣}$

Orthogonal vectors:

The two vectors $\vec{A}$ and $\vec{B}$ are orthogonal if and only if:

$\vec{A}.\vec{B} = 0$

Lecture 5: Partial Derivatives

1. Introduction

In many real-life applications, functions depend on more than one variable such as f(x, y) or f(x, y, z). To understand how the function changes when only one variable changes while all others remain constant, we use partial derivatives.

2. Definition

If f(x, y) is a function of two variables, then the partial derivative with respect to x is:

$$\frac{\partial f}{\partial x}$$

Meaning: differentiate with respect to x while treating y as a constant.

Similarly,

$$\frac{\partial f}{\partial y}$$

is the derivative with respect to y.

Example 1:

$f(x, y) = x^2 y + 5y$

$\frac{\partial f}{\partial x} = 2xy$

$\frac{\partial f}{\partial y} = x^2 + 5$

Example 2:

$f(x, y) = \sin(x y)$

$\frac{\partial f}{\partial x} = \cos(xy) * y$

$\frac{\partial f}{\partial y} = \cos(x y) * x$

Example 3:

$f(x, y, z) = x^2 + yz + e^{xz}$

$f_x = 2x + z e^{xz}$

$f_y = z$

$f_z = y + x e^{xz}$

Ex.: Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of the following functions:

a. $z = x^2 + 3xy + y - 1$

b. $z = \ln(x^2 - y)$

c. $z = x\cos(y) + ye^x$

d. $z = y\sin(xy)$

Sol.:

a. To find $\frac{\partial z}{\partial x}$ treat y as a constant and differentiate with respect to x. We have $z = x^2 + 3xy + y - 1$ so:

$\frac{\partial z}{\partial x} = 2x + 3y$

Similarly

$\frac{\partial z}{\partial y} = 3x + 1$

b. $z = \ln(x^2 - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{(x^2 - y)} 2x = \frac{2x}{(x^2 - y)} \quad , \quad \frac{\partial z}{\partial y} = \frac{-1}{(x^2 - y)}$$

c. $z = x\cos(y) + ye^x$

$$\frac{\partial z}{\partial x} = \cos(y) + ye^x \quad , \quad \frac{\partial z}{\partial y} = -x\sin(y) + e^x$$

d. $z = y\sin(xy)$

$$\frac{\partial z}{\partial x} = y\cos(xy)y = y^2\cos(xy)$$

$$\frac{\partial z}{\partial y} = xy\cos(xy) + \sin(xy)$$

2- Derivatives of the Natural Logarithm Function (ln)

1. Derivative of ln(x) in One Variable

$$d/dx \ln(x) = \frac{1}{x} \quad , \quad x > 0$$

- Example:

$$\frac{d}{dx} \ln(3x^2 + 1) = (1 / (3x^2 + 1)) * 6x = \frac{6x}{(3x^2 + 1)}$$

3- Partial Derivative of a Multivariable ln Function

If we have $f(x, y) = \ln(xy + y^2)$, the partial derivatives are:

1. $\partial f/\partial x$

$f = \ln(u)$, $u = xy + y^2$

$\partial f/\partial x = 1/u * \partial u/\partial x = 1/(xy + y^2) * y = y / (xy + y^2)$

4- More Complex ln Function

Example: $f(x, y, z) = \ln(xy + z^2)$

Partial derivative w.r.t x: $\partial f/\partial x = 1/(xy + z^2) * y = y / (xy + z^2)$

Partial derivative w.r.t y: $\partial f/\partial y = 1/(xy + z^2) * x = x / (xy + z^2)$

Partial derivative w.r.t z: $\partial f/\partial z = 1/(xy + z^2) * 2z = 2z / (xy + z^2)$

5- Derivative of e^x in One Variable

$d/dx\ e^x = e^x$

- Basic rule: the derivative of e^x is itself.

- Example:

$d/dx\ e^{3x} = 3\ e^{3x}$

Partial Derivative of a Multivariable Exponential Function

If we have $f(x, y) = e^{xy + y2}$, the partial derivatives are:

1. ∂f/∂x

$f = e^u$, $u = xy + y^2$

$\partial f/\partial x = e^u * \partial u/\partial x = e^{xy + y2} * y = y\ e^{xy + y2}$

∂f/∂y

$\partial u/\partial y = x + 2y$

$\partial f/\partial y = e^u * (x + 2y) = (x + 2y)\ e^{xy + y2}$

More Complex Exponential Functions

Example: $f(x, y, z) = e^{xy + z2}$

Partial derivative w.r.t x: $\partial f/\partial x = e^{(xy + z2)} * y = y\ e^{(xy + z2)}$

Partial derivative w.r.t y: $\partial f/\partial y = e^{(xy + z2)} * x = x\ e^{(xy + z2)}$

Partial derivative w.r.t z: $\partial f/\partial z = e^{(xy + z2)} * 2z = 2z\ e^{(xy + z2)}$